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A model for rank one measure preserving transformations ☆

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ABSTRACT

We define a model for rank one measure preserving transformations in the sense of [2]. This is done by defining a new Polish topology on the space of codes, which are infinite rank one words, for symbolic rank one systems. We establish that this space of codes has the same generic dynamical properties as the space of (rank one) measure preserving transformations on the unit interval.

1. Introduction

There are several definitions of rank one measure preserving transformations in the literature (cf. a summary in [1]). They are generally considered equivalent, and none are easy to give. Among them, the two most useful have been the constructive geometric definition and the constructive symbolic definition. According to the constructive geometric definition, a rank one transformation is a measure preserving transformation of the unit interval that is obtained by a cutting and stacking process. The constructive symbolic definition, however, defines a rank one system as a special kind of Bernoulli subshift.

In ergodic theory it is important to speak of a generic dynamical property of measure preserving transformations. In order to do this one needs to fix a topology on the space of all measure preserving

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transformations. Let **MPT** denote the collection of all invertible measure preserving transformations on the unit interval with the Lebegue measure (by identifying transformations that agree on a set of measure one). Then the weak topology on **MPT** is Polish and is considered the standard topology.

Let $\mathbf{R}_1$ be the subcollection of **MPT** consisting of all rank one transformations, i.e., transformations that are isomorphic to constructive rank one transformations. Then $\mathbf{R}_1$ is a dense G_δ subset of **MPT**. Thus $\mathbf{R}_1$ is not only a Polish space in its own right with the subspace topology inherited from **MPT**, but also a generic class in **MPT**. In particular, a dynamical property of rank one transformations is generic iff it is generic as a property for all measure preserving transformations.

The situation on the side of the constructive symbolic definition of rank one systems, however, is not clear. Recently the authors defined in [3] a natural space $\mathcal{R}$ of codes for all symbolic rank one homeomorphisms. We also showed that it has a natural Polish topology. Thus it makes sense to consider the subspace $\mathcal{R}^*$ of all codes for symbolic systems that correspond to rank one measure preserving systems. Unfortunately $\mathcal{R}^*$ is neither generic in $\mathcal{R}$ nor Polish with the subspace topology. Thus in order to speak of generic dynamical properties on $\mathcal{R}^*$ we need to redefine a Polish topology on $\mathcal{R}^*$.

As $\mathcal{R}^*$ is a Borel subset of $\mathcal{R}$ such Polish topology on $\mathcal{R}^*$ definitely exists. However, we would like the topology on $\mathcal{R}^*$ to have better properties. First, it is desirable that the topology on $\mathcal{R}^*$ is naturally connected to its meaning as the space of all rank one measure preserving systems. Second, it would be nice if with an appropriate definition of the topology on $\mathcal{R}^*$ the two spaces $\mathcal{R}^*$ and $\mathbf{R}_1$ have the same generic dynamical properties. Moreover, we want an explicit correspondence between $\mathcal{R}^*$ and $\mathbf{R}_1$ that would map a code in $\mathcal{R}^*$ to a generic transformation in $\mathbf{R}_1$ isomorphic to the coded system.

We achieve all these in this paper. We will define a natural Polish topology on $\mathcal{R}^*$ and prove that $\mathcal{R}^*$ and $\mathbf{R}_1$ share the same generic dynamical properties. This is done by making use of the concept of a *model* defined by Foreman, Rudolph and Weiss in [2]. The main theorem of the current paper is to show that $\mathcal{R}^*$ is a model of (rank one) measure preserving transformations in the sense of [2]. That $\mathcal{R}^*$ and $\mathbf{R}_1$ share the same generic dynamical properties is a corollary of the main theorem.

Since many interesting results about rank one transformations were proved by combinatorial methods applied to the symbolic context, the results established in this paper are potentially useful for further studies of generic behavior of rank one transformations.

2. The standard model

In this section we recall the basic definitions and establish the standard model for rank one measure preserving transformations.

A *measure preserving system* is a quadruple $(X, \mathcal{B}, \mu, T)$ where X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X, μ a separable non-atomic probability measure on $\mathcal{B}$, and T an invertible μ-preserving transformation, i.e., $T : X \to X$ is an invertible map such that for all $A \in \mathcal{B}$, $T^{-1}(A) \in \mathcal{B}$ and $\mu(T^{-1}(A)) = \mu(A)$.

In the above definition the triple $(X, \mathcal{B}, \mu)$ is called a *Lebegue space*. It is well known that any Lebesgue space is isomorphic, modulo a null set, to the unit interval $[0, 1]$ with the σ-algebra of all Borel sets and the standard Lebesgue measure λ. For notational simplicity we denote this canonical Lebesgue space by just $[0, 1]$. It follows that any measure preserving system is isomorphic to one on $[0, 1]$ with some Lebesgue measure preserving $T : [0, 1] \to [0, 1]$. It is thus natural to consider the collection

$$\mathbf{MPT} = \{T : [0,1] \to [0,1] \mid T \text{ is Lebesgue measure preserving}\},$$

with identification of T and S if

$$\lambda(\{x \in [0,1] \mid T(x) = S(x)\}) = 1,$$