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An internal characterisation of radiality

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ABSTRACT

In this paper, we will investigate how radiality occurs in topological spaces by considering neighbourhood bases generated by nests. We will define a new subclass of radial spaces that contains LOTS, GO-spaces and spaces with well-ordered neighbourhood bases, called the independently-based spaces. We show that first-countable spaces are precisely the independently-based, strongly Fréchet spaces and we give an example of a Fréchet–Urysohn space that is neither independently-based nor strongly Fréchet.

1. Introduction

Radial spaces were first introduced in [1] under the name of *stark folgenbestimmt*, and were characterised in that paper as the pseudo-open images of LOTS (*Linearly Ordered Topological Spaces*). They are a natural generalisation of Fréchet–Urysohn spaces. Although we have an external characterisation of these spaces via certain quotients of LOTS, the author felt it was insufficient to truly gain an appropriate understanding for these spaces. Thus, the author wished to find an *internal* characterisation that would lead to a deeper understanding of radiality.

The most common examples of radial spaces are LOTS, GO-spaces (*Generalised Ordered* spaces) and spaces with well-ordered neighbourhood bases (e.g., first-countable spaces). These can be viewed as having

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neighbourhoods generated by nests (sets linearly ordered by inclusion). It is these spaces which we shall generalise from to create the class of *independently-based* spaces. These spaces have a clear picture of convergence and although they don't coincide with radial spaces (as we will show), a slight weakening of the definition does give such a characterisation of radiality in terms of a type of neighbourhood base generated by subspaces that have well-ordered neighbourhood bases at the point in question. Briefly, these neighbourhood generators describe the 'paths of convergence' to the point.

In the next section, we will introduce some terminology and basic results we shall need. In Section 3, we will introduce the class of spaces called the independently-based spaces that have neighbourhoods generated by nests in an appropriate way. We will show that all GO-spaces and *well-based* spaces (spaces with well-ordered neighbourhood bases) are independently-based and that independently-based spaces are radial. Moreover, the property of being independently-based is hereditary. In Section 4, we will give a characterisation of radiality using a different kind of neighbourhood generator and prove that first-countable spaces are precisely the independently-based, strongly Fréchet spaces. To finish with, we will construct a Fréchet–Urysohn space that is neither independently-based nor strongly Fréchet.

We will not be assuming any separation axioms in this paper. For more on radial spaces and related convergence properties, see [2–4].

2. Preliminaries

Below are some definitions that are required for this paper.

Definition 2.1.

- A *net* is a function f from a directed set $(D, \leq)$ to a set X. If X is a (topological) space then we say that f *converges to* $x \in X$, denoted by $f \to x$, if for every neighbourhood U of x, there exists $d \in D$ such that for all $d' \geq d$, $f(d') \in U$. It is said to *cluster* at x if for all neighbourhoods U of x and $d \in D$, there exists a $d' \geq d$ such that $f(d') \in U$.
- A *transfinite sequence* is a net with well-ordered domain. We will use the notation $(x_\alpha)_{\alpha<\lambda}$ for the transfinite sequence f with domain $(\lambda, \in)$, where λ is an ordinal and for all $\alpha < \lambda$, $f(\alpha) = x_\alpha$.
- We say that a (topological) space X is *radial* at a point x if for every subset A of X that contains x in its closure, there is a transfinite sequence converging to x whose range lies in A. If a space is radial everywhere then we call it a *radial* space.
- A space X is *strongly Fréchet* at a point x if for every decreasing sequence $(A_n)_{n<\omega}$ of subsets of X with $x \in \bigcap_{n<\omega} \overline{A_n}$, there exists a sequence $(x_n)_{n<\omega}$ that converges to x such that $x_n \in A_n$ for all $n < \omega$. If a space is strongly Fréchet everywhere then we call it a *strongly Fréchet* space.
- A space X is said to be *well-based* at x if x has a neighbourhood base well-ordered by $\supseteq$. Such a neighbourhood base is said to be *well-ordered neighbourhood base* and if X is well-based at every point it is called a *well-based* space.
- A transfinite sequence is said to *converge strictly* to a point x in a space if it converges to x and x is not in the closure of any of the initial segments of the transfinite sequence.
- A *nest* is a non-empty set linearly ordered by inclusion.
- For a linearly ordered set $(X, <)$, we define its *cofinality* to be the least ordinal α such that there exists a cofinal map $f : (\alpha, \in) \to (X, <)$. This will be denoted by $\mathrm{cf}(X, <)$.
- For a point x in a space X, we denote its neighbourhood filter by $\mathcal{N}_x^X$, or $\mathcal{N}_x$ when the space is unambiguous. We define its *neighbourhood core* to be the intersection of all neighbourhoods of x. This will be denoted by N_x. Note that in a T_1-space, $N_x = \{x\}$.
- A point x in a space X is *quasi-isolated* if N_x is open (or equivalently, N_x is a neighbourhood of x).