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On the structure of fundamental groups of conic-line arrangements having a cycle in their graph

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A R T I C L E I N F O

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A B S T R A C T

The fundamental group of the complement of a plane curve is a very important topological invariant. In particular, it is interesting to find out whether this group is determined by the combinatorics of the curve or not, and whether it is a direct sum of free groups and a free abelian group, or it has a conjugation-free geometric presentation.
In this paper, we investigate the structure of this fundamental group when the graph of the conic-line arrangement is a unique cycle of length n and the conic passes through all the multiple points of the cycle. We show that if n is odd, then the affine fundamental group is abelian but not conjugation-free. For the even case, if $n > 4$, then using quotients of the lower central series, we show that the fundamental group is not a direct sum of a free abelian group and free groups.
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1. Introduction

The fundamental group of the complement of a plane curve is a very important topological invariant. For example, it is used to distinguish between curves that form a Zariski pair, which is a pair of curves having the same combinatorics but non-homeomorphic complements in $\mathbb{CP}^2$ (see [4] for the exact definition and [5] for a survey). Moreover, the Zariski–Lefschetz hyperplane section theorem (see [20]) states that $\pi_1(\mathbb{CP}^N - S) \cong \pi_1(H - (H \cap S))$, where S is a hypersurface in $\mathbb{CP}^N$ and H is a generic 2-plane. Since $H \cap S$ is a plane curve, the fundamental groups of complements of plane curves can also be used for computing the fundamental groups of complements of hypersurfaces. Note that when S is a hyperplane arrangement, $H \cap S$ is a line arrangement in $\mathbb{CP}^2$. Thus, one of the main tools for investigating the topology of hyperplane arrangements is the fundamental groups $\bar{G} = \pi_1(\mathbb{CP}^2 - \mathcal{L})$ and $G = \pi_1(\mathbb{C}^2 - \mathcal{L})$, where $\mathcal{L}$ is an arrangement of lines.

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One of the main questions arising in the research of hyperplane arrangements is how does the combinatorics – in this case, the intersection lattice – of such an arrangement determine the fundamental group G or the quotients, for example, of its lower central series G_i/G_{i+1} (where $G_1 = G$ and $G_i = [G_{i-1}, G]$). For example, when does the arrangement have a conjugation-free geometric presentation for its fundamental group? Also, it is well-known that for line arrangements, G/G_2, G/G_3 and G_2/G_3 are determined by the combinatorics (see Section 2.3) and in fact Falk [10] has shown that the rank of the quotients G_i/G_{i+1} is also determined by the combinatorics. However, as Rybnikov shows [23], the quotient G/G_4 is not determined by the combinatorics, at least for complex arrangements.

These questions lead us to investigate the situation in the simplest generalization of arrangements of lines: conic-line (CL) arrangements. Indeed, some families of CL arrangements were studied by Amram et al. (see e.g. [1,2] and especially [3, Theorem 6]). We also showed in [13] that the combinatorics of some families of real CL arrangements determines the structure of the corresponding fundamental group G and that G is conjugation-free. However, the family $\mathcal{A}_n$, where the graph of the arrangement is a cycle of length n and the conic passes through all the vertices of the graph, poses problems: not only that these arrangements are not conjugation-free (at least for odd n), but one has to differentiate between cycles whose lengths have different parity.

In this paper, we give a complete description of the affine fundamental group $\pi_1(\mathbb{C}^2 - \mathcal{A}_n)$ for the case of odd n: in this case, the fundamental group is abelian but not conjugation-free. For the case of even n, we prove that the fundamental group is not abelian and not a direct sum of a free abelian group and free groups. The last statement is proven by studying the groups G_2/G_3, G/G_3 and $Z(G/G_3)$.

The paper is organized as follows. In Section 2, we survey the known results on line arrangements, the conjugation-free property and certain quotients of the fundamental group arising from the lower central series. In Section 3, we examine two special cases, when the arrangement is as above and the cycle is of length 3 or 4, and in Section 4 we prove the main result: while for odd n, the fundamental group is abelian and not conjugation-free, for even $n > 4$, the fundamental group is not a sum of a free abelian group and free groups.

2. Arrangements and the conjugation-free property

In this section, we give a short survey of the known results concerning the structure of the fundamental group of the complement of line arrangements and conic-line arrangements, while mentioning also the conjugation-free property.

2.1. Arrangements and their associated graphs

An *affine line arrangement* in $\mathbb{C}^2$ is a union of copies of $\mathbb{C}^1$ in $\mathbb{C}^2$. Such an arrangement is called *real* if the defining equations of all its lines can be written with real coefficients, and *complex* otherwise.

For a real or complex line arrangement $\mathcal{L}$, Fan [12] defined a graph $G(\mathcal{L})$ which is associated to the multiple points of $\mathcal{L}$ (i.e. points where more than two lines are intersected). We give here its version for real arrangements (the general version is more delicate to explain and will be omitted): Given a real line arrangement $\mathcal{L}$, the graph $G(\mathcal{L})$ lies on the real part of $\mathcal{L}$. Its vertices are the multiple points of $\mathcal{L}$ and its edges are the segments between the multiple points on lines which have at least two multiple points. Note that if the arrangement consists of three multiple points on the same line, then $G(\mathcal{L})$ has three vertices on the same edge (see Fig. 1(a)). If two such lines happen to intersect in a *simple* point (i.e. a point where exactly two lines are intersected), it is ignored (i.e. there is no corresponding vertex in the graph). See another example in Fig. 1(b) (note that Fan's definition gives a graph slightly different from the graph defined in [16,24]).

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