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Algebraically compact rings

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ABSTRACT

We introduce the notion of algebraically compact rings and prove that central results of theory of compact rings can be extended to this settings.

1. Introduction

The following problem is central in the theory of topological rings: What relations exist between algebraic properties of a ring R and ring topologies which can be introduced on it?

Recall that an Abelian group is called *algebraically compact* provided it is a direct summand of a compact Abelian group [6]. Kaplansky ([9], p. 56; [6], Theorem 39.1) has proved that a reduced group is algebraically compact if an only if it is complete in the Z-adic topology.

Recall that a subring S of a ring R is called a *retract of* R if there exists a homomorphism $f : R \to S$ such that $f(s) = s$ for all $s \in S$.

A topological ring $(R, \mathcal{T})$ is called *algebraically compact* if its completion $(cR, c\mathcal{T})$ is compact and R is a retract in the algebraic sense of cR.

Remark 1.1. It follows from the uniqueness of completions of topological rings that a topological ring $(R, \mathcal{T})$ is algebraically compact if and only if there exists a compact ring R' such that R is a retract of it and topology of R is induced by topology of R'.

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Clearly, every compact ring is algebraically compact. We will prove that any semisimple algebraically compact ring is a retract of a compact semisimple ring and its topology is the induced one.

Boolean algebraically complete rings are exactly totally bounded topological rings complete in the sense of the theory of Boolean algebras.

It was proved in [11] that a compact nilring is a ring of finite nil index. In particular, a torsion free compact associative nilring is nilpotent. We prove here that every torsion free associative nilring whose additive group is a reduced algebraically compact group in the algebraic sense is nilpotent. We prove also that in any algebraically compact ring the Levitzki radical coincides with the Kőthe radical.

2. Notation and conventions

All rings are assumed associative, but not necessarily with identity. Topological rings and groups are assumed to be Hausdorff. The Jacobson radical of a ring R is denoted by $J(R)$. A *semisimple* ring means semisimple with respect to the Jacobson radical. By $\mathbb{F}_2$ we denote the field consisting of two elements. The symbol R_0 stands for the connected component of zero of a topological ring R. We denote by $cl_{(X,\mathcal{T})}(A)$ the closure of a subset A of a topological space $(X,\mathcal{T})$. If X is a subset of a ring R, then $\langle X\rangle$ stands for the subring of R generated by X. A *totally bounded* topological ring is a ring whose completion is compact. If A is a subset of a ring R then by A^* we denote the subset $\{x \in R \mid xA = Ax = 0\}$. Recall that the *Bohr topology* on a ring [2] is the finest totally bounded topology on it. If R is a Boolean ring, then the set of all ideals of finite index is a fundamental system of neighborhoods of zero for the topological ring $(R,\mathcal{T}_{Bohr})$, where $\mathcal{T}_{Bohr}$ is the Bohr topology on R. The completion of a topological ring $(R,\mathcal{T})$ is denoted by $(cR, c\mathcal{T})$. A topological ring is called *linear* if it has a fundamental system of neighborhoods of zero consisting of ideals. A *regular* ring is a ring regular in the sense of von Neumann. A topological ring is called *monocompact* (*topologically locally finite*) if each of its element (each of its finite subsets) is contained in a compact subring [12].

3. Preliminaries and some examples

The following assertions follow immediately:

1. The class of algebraically compact rings is multiplicative.
2. Matrix rings over algebraically compact rings are algebraically compact.
3. The additive group of an algebraically compact ring is algebraically compact in the sense of the theory of Abelian groups.
4. The ring $R = \mathbb{F}_2^\omega$ has two ring topologies $\mathcal{T}_1, \mathcal{T}_2$ such that $(R,\mathcal{T}_1)$, $(R,\mathcal{T}_2)$ are non-isomorphic algebraically compact rings.

Indeed, consider on the ring R the compact ring topology $\mathcal{T}_1$ and a dense maximal ideal V of $(R,\mathcal{T}_1)$. Consider the ring topology $\mathcal{T}_2$ on R having the family of ideals $U \cap V$ where U runs all open ideals of $(R,\mathcal{T}_1)$ as a fundamental system of neighborhoods of zero. Obviously, $\mathcal{T}_1 \le \mathcal{T}_2$ and $\mathcal{T}_1 \ne \mathcal{T}_2$.

The ring $(R,\mathcal{T}_2)$ is algebraically compact. Indeed, let $(R/V) \times R$ be the product of the discrete ring R/V and $(R,\mathcal{T}_1)$. Consider the embedding $h : R \to (R/V)\times R, r \mapsto (r+V)\times r$. Since $h^{-1}[(\{0\}\times U)\cap h(R)] = U\cap V$ for every ideal U of R, h is a topological embedding and we can identify $(R,\mathcal{T}_2)$ with the subring $h(R)$ of $(R/V) \times R$. Furthermore, since $h(R) + (R/V) \times \{0\} = (R/V) \times R$ and $h(R) \cap (R/V) \times \{0\} = 0$, $h(R)$ is a retract of $(R/V) \times R$. Therefore the ring $(R,\mathcal{T}_2)$ is algebraically compact.

5. The ring $\mathbb{Z}$ does not admit algebraically compact ring topologies.
6. The factor ring of an algebraically compact ring is not necessarily algebraically compact.