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A construction of the B-completion of a T_0-quasi-metric space

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A B S T R A C T

We present a construction of the B-completion of a T_0-quasi-metric space (X, d) that is based on the injective hull of (X, d). The B-completion of a T_0-quasi-metric space (X, d) is a T_0-quasi-metric extension of (X, d) studied by H.-P.A. Künzi and C. Makitu Kivuvu.

In general, the B-completion of a T_0-quasi-metric space is larger than the better known bicompletion of (X, d). In the special case of a balanced T_0-quasi-metric space the B-completion coincides with a completion that was first investigated by D. Doitchinov.

Among other things, in this note we show that the B-completion of the natural T_0-quasi-metric of a partially ordered set (possessing a smallest and a largest element) can be identified with the Dedekind–MacNeille completion of that partially ordered set.

1. Introduction

Under the name of the B-completion H.-P.A. Künzi and C. Makitu Kivuvu developed a quasi-metric completion theory for an arbitrary T_0-quasi-metric space [14,15]. The B-completion of a T_0-quasi-metric space is sometimes helpful in cases where the bicompletion turns out to be unnaturally small. For instance, if we equip the set of the rationals with the usual Sorgenfrey T_0-quasi-metric the corresponding space is bicomplete, while — more naturally — the B-completion of that space can be identified with the set of the reals equipped with the Sorgenfrey T_0-quasi-metric [17, Example 5]. Applied to a balanced T_0-quasi-metric space the B-completion indeed yields a completion that was introduced by Doitchinov [5,14] (compare also [6,16]).

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The theory of the B-completion due to H.-P.A. Künzi and C. Makitu Kivuvu is "quasi-metric" and not "quasi-uniform" in the sense that two quasi-metrics may induce the same quasi-uniformity on a set, with one being B-complete, while the other is not (see [15, Example 6]).

Recently the injective hull (or q-hyperconvex hull) of a T_0-quasi-metric space X was studied by various authors (see e.g. [9,12,22], and [7,11] for corresponding metric investigations). Categorically, that construction can be understood as a kind of Dedekind–MacNeille completion of T_0-quasi-metric spaces (compare [1,22]). In particular the injective hull of a T_0-quasi-metric space (X,d) contains the bicompletion of (X,d) as a subspace. Since there are obvious connections between Doitchinov's idea of completing T_0-quasi-metric spaces and the Dedekind–MacNeille completion of partially ordered sets (compare e.g. [2,3] or Proposition 3 below), it is natural to wonder whether in fact the B-completion of a T_0-quasi-metric space also sits inside its injective hull. In this note we shall verify that conjecture and illustrate some of the connections between the B-completion of T_0-quasi-metric spaces and the Dedekind–MacNeille completion of partially ordered sets.

2. Preliminaries

In this section we recall some of the basic terminology used throughout this note.

Let X be a set and let $d : X \times X \to [0,\infty)$ be a function mapping into the set $[0,\infty)$ of the non-negative reals. Then d is called *a quasi-pseudometric* on X if (a) $d(x,x) = 0$ whenever $x \in X$, and (b) $d(x,z) \leqslant d(x,y) + d(y,z)$ whenever $x,y,z \in X$.

We say that d is a T_0*-quasi-metric* if d also satisfies the following condition: For each $x,y \in X$, $d(x,y) = 0 = d(y,x)$ implies that $x = y$.

If we set, for $x,y \in X$, $x \leqslant_d y$, if and only if $d(x,y) = 0$, then $\leqslant_d$ is the so-called *specialization partial order* of d.

By $\mathcal{U}_d$ we shall denote the *quasi-uniformity induced by* d on X.

Furthermore for any $x \in X$, $\mathcal{U}_d(x)$ will denote the filter on X generated by the base $\{B_d(x,\epsilon)\colon \epsilon > 0\}$ where $B_d(x,\epsilon) = \{y \in X\colon d(x,y) < \epsilon\}$, $\epsilon > 0$. For concepts from the theory of quasi-uniformities we refer the reader to [8,13].

Given two real numbers a and b we shall write $a \dot{-} b$ for $\max\{a-b,0\}$. By $\mathbb{N}$ we shall denote the set of natural numbers $\{1,2,\ldots\}$.

Let d be a quasi-pseudometric on X, then $d^{-1} : X \times X \to [0,\infty)$ defined by $d^{-1}(x,y) = d(y,x)$ whenever $x,y \in X$ is also a quasi-pseudometric, called the *conjugate quasi-pseudometric of* d. Note that $d^s = \max\{d,d^{-1}\}$ is a metric if d is a T_0-quasi-metric.

A map $f : (X,d) \to (Y,e)$ between two quasi-pseudometric spaces (X,d) and (Y,e) is called an *isometry* provided that $e(f(x),f(y)) = d(x,y)$ whenever $x,y \in X$. Two quasi-pseudometric spaces (X,d) and (Y,e) will be called *isometric* provided that there exists a bijective isometry $f : (X,d) \to (Y,e)$.

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in a set X. Moreover let $\mathcal{F}(x_n)$ be the filter generated by the filter base $\{\{x_k\colon k \geqslant n, k \in \mathbb{N}\}\colon n \in \mathbb{N}\}$ on X. Then we shall say that $\langle\mathcal{F}(x_n),\mathcal{F}(y_n)\rangle$ *is the filter pair on* X *generated by the pair* $\langle(x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}}\rangle$ *of sequences in* X.

Let $\mathcal{F} = \langle\mathcal{F}_1,\mathcal{F}_2\rangle$ and $\mathcal{F}' = \langle\mathcal{F}'_1,\mathcal{F}'_2\rangle$ be two filter pairs on a set X. Then $\langle\mathcal{F}_1,\mathcal{F}_2\rangle$ is called *coarser* than $\langle\mathcal{F}'_1,\mathcal{F}'_2\rangle$ provided that both $\mathcal{F}_1 \subseteq \mathcal{F}'_1$ and $\mathcal{F}_2 \subseteq \mathcal{F}'_2$.[1]

[1] Following the notations of [14] resp. [12], in this paper a filter pair $\mathcal{F} = \langle\mathcal{F}_1,\mathcal{F}_2\rangle$ will always be denoted as the pair $\langle\mathcal{F}_1,\mathcal{F}_2\rangle$, while — somewhat inconsistently — a pair of functions $f = (f_1,f_2)$ will often be denoted by the single letter f.