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The comparison of topologies related to various concepts of generalized covering spaces

Žiga Virk [a,*], Andreas Zastrow [b]

[a] *Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia*
[b] *Institute of Mathematics, Gdansk University, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland*

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A B S T R A C T

The most common construction of a generalized covering space is that of a topologized (appropriate version of a) path space $\widetilde{X}$. There have been three suggested topologies on it, each with its advantages and disadvantages. They are called the Whisker topology, the Lasso topology and the quotient of the compact open topology. In this paper we study the relationship between these topologies. The main result consists of an example demonstrating that the Lasso topology is not finer that the compact open topology.
Our results also apply to the topology of the fundamental group which appears naturally as a subspace of $\widetilde{X}$.

1. Introduction

The classical theory of covering spaces is based on the construction of the universal covering space of the space X as a space of homotopy types of certain paths. In the case of path connected, locally path connected, semilocally simply connected spaces there is an obvious choice of topology on such space, which makes it the universal covering space. However, when faced with more general spaces the choice of topology becomes more challenging.

One of the possible topologies is the Whisker topology (see Definition 2), which had been used in [8, the proof of Theorem 13, Section 5, Chapter 2], [2, Section 2.1] and [7, Section 2] before being named in [3]. In this topology a neighborhood of a homotopical class of a path consists of all paths obtained by prolonging the original path by a small amount.

The Whisker topology is perhaps the most obvious choice of topology and allows a certain generalization of the classical covering space theory. However, it possesses an asymmetry in the essence of its definition which made certain analogies of the classical results impossible. In order to overcome the corresponding

* Corresponding author.
E-mail addresses: zigavirk@gmail.com (Ž. Virk), zastrow@mat.ug.edu.pl (A. Zastrow).

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difficulties the Lasso topology (see Definition 4) has been introduced and studied in [3]. In this topology a neighborhood of a homotopical class of a path consists of all paths obtained by modifying the original path in the following ways: by inserting sufficiently small loops at any point of the path and by prolonging the original path by a small amount. An important aspect of this topology is that the fundamental group, which naturally appears as a subspace of the homotopy classes of paths, becomes a topological group when equipped with the topology induced by the Lasso topology.

The third topology is the quotient topology, induced by the compact open topology on the space of pointed paths. For the sake of simplicity we will call it the CO-topology (see Definition 5). It was introduced in [1] in an attempt to topologize the fundamental group which appears naturally as a subspace of our space.

The aim of this paper is to clarify the relationship between these three topologies. A partial comparison was given in [3, Propositions 3.6 and 3.7]. For the sake of completeness we list the past results together with the results of this paper:

(1) The difference between the topologies is manifested in the example of the Hawaiian earring $\mathbb{H}\!\!\mathbb{E}$. It turns out [3, Propositions 3.6 and 3.7] that the topologies differ in the case of the fundamental group $\pi_1(\mathbb{H}\!\!\mathbb{E})$. The topologized $\pi_1(\mathbb{H}\!\!\mathbb{E})$ is a topological group only when equipped with the Lasso topology. (See also [4,5].)
(2) If the space is path connected, locally path connected and semilocally simply connected then the topologies coincide with the classical universal covering space.
(3) It is apparent that the Whisker topology is in general strictly finer than the Lasso topology.
(4) In this paper we prove that the Lasso topology and the CO-topology are incomparable: neither is (in general) finer than the other. (See Theorems 6 and 14.)
(5) We prove that the CO topology is finer than the Lasso topology for locally path-connected spaces. (See Theorem 9.) This observation was independently made by Hanspeter Fischer [6].
(6) We also note (see [7, Lemma 2.1] for the first proof of this fact) that the Whisker topology is in general strictly finer than the CO topology. (See the discussion at the end of Section 2 and Proposition 7.)

2. Preliminaries

Throughout the paper I denotes the closed unit interval in $\mathbb{R}$ and (X, x_0) is a pointed path-connected space. Given a path $\gamma\colon [a,b] \to X$ its *reversed path* is denoted by $\gamma^-\colon [a,b] \to X$ and defined as $\gamma^-(a+t) = \gamma(b-t)$, $\forall t \in [0, b-a]$. In particular this implies, that for paths parametrized on the unit interval we have that $\gamma^-(t) = \gamma(1-t)$. However, since this parameter transformation is sometimes confusing, we hereby agree, that if we use the notation of the restriction of a reversed path to a subsegment, we always refer to the parameters of the original path. I.e. we understand

$$\gamma^-|_{[t_0,t_1]} \text{ as an abbreviation for } (\gamma|_{[t_0,t_1]})^-.$$

Definition 1. Given a pointed topological space (X, x_0) we define PX as the collection of compact paths in X whose starting point is x_0:

$$PX = \{\alpha\colon I \to X \mid \alpha(0) = x_0\}.$$

Space $\widetilde{X}$ is obtained from PX by identifying paths that are homotopic (relatively to the endpoints).

There are three topologies that have been applied to $\widetilde{X}$. In the case of path-connected, locally path-connected, semilocally simply connected spaces all three topologies correspond to the usual topology on the universal covering space $\widetilde{X}$.