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On non-metric continua that support Whitney maps

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ABSTRACT

We discuss the construction of non-metric continua that support Whitney maps and their properties. We indicate a technique for producing non-metric indecomposable continua that support Whitney maps from certain compact totally disconnected spaces each of which allows a self-homeomorphism all of whose orbits are dense. These are non-metric examples that have the property that each proper subcontinuum is metric. Both perfectly normal and non-perfectly normal examples are constructed. We describe techniques for producing large collections of non-homeomorphic continua that support Whitney maps. An example of a continuum every non-degenerate subcontinuum of which is non-metric that supports a Whitney map is constructed; an example of a continuum that does not support a Whitney map which is the union of two subcontinua each of which supports a Whitney map is constructed.

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1. Introduction

Suppose X is a topological space. Let 2^X denote the space of compact subsets of X with the Vietoris topology. We let $C(X)$ denote the subspace of 2^X consisting of the subcontinua of X. A Whitney map μ is a continuous function $\mu : 2^X \to \mathbb{R}$ that has the property that for $x \in X$, $\mu(\{x\}) = 0$, and for $H \subsetneq K \in 2^X$, $\mu(H) < \mu(K)$. Suppose that X is a compact Hausdorff space that supports a Whitney map μ. Then the function $f : X \times X \to \mathbb{R}$ defined by $f(x, y) = \mu(\{x, y\})$ is continuous and identically 0 on the diagonal of $X \times X$ so, by the continuity of f, the diagonal is a G_δ set. So it follows from the result of Šneĭder [9] that X is metric. So a non-metric space does not support Whitney maps on its hyperspace 2^X. However, J. Charatonik and W. Charatonik [3] gave an example of a non-metric continuum X and Whitney map that is restricted to the hyperspace $C(X)$. This example has the property that each of its proper subcontinua is metric. Based on an example of a continuum that appears to be homeomorphic to the continuum constructed by J. Charatonik and W. Charatonik, one of us [Stone] in her dissertation [10] constructed a continuum

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that has the property that also has the property and which we believe is homeomorphic to the example of J. Charatonik and W. Charatonik. We describe a technique for producing other continua with this property. The example of J. Charatonik and W. Charatonik appears to be perfectly normal and we construct examples of non-perfectly normal continua that support Whitney maps, including one each proper subcontinuum of which is metric. Furthermore we construct a continuum that supports a Whitney map each non-degenerate subcontinuum of which is non-metric. This continuum is not perfectly normal.

Definition. The continuum X is said to support a Whitney map on its set of subcontinua $C(X)$ if there is a continuous function $\mu : C(X) \to \mathbb{R}$ so that:

1. If $x \in X$ then $\mu(\{x\}) = 0$;
2. If $H, K \in C(X)$ and $H \subsetneq K$ then $\mu(H) < \mu(K)$.

In this paper, if X is non-metric then the statement "X supports a Whitney map" will mean that X supports a Whitney map on $C(X)$. Extensive discussions of Whitney maps in the metric setting is available in Nadler [7].

2. Construction of non-metric continua that support Whitney maps

Definition. Let $F : Z \to Z$ be a function, then F is said to have the *block permutation property* means that for each non-empty open set $U \subset Z$ and $t \in U$ there is an open set S with $t \in S \subset U$ and an integer N_S so that $\{F^n(S)\}_{n=0}^{N_S}$ is a disjoint collection of open sets covering Z.

Construction 1. Suppose that Z is an infinite compact totally disconnected Hausdorff space and $F : Z \to Z$ is a homeomorphism that has the block permutation property. Suppose further that I is a Hausdorff arc with end points a and b, $X = Z \times I$ and that G is the upper semi-continuous decomposition of X that identifies the points (z, b) with the point $(F(z), a)$. Then let Y denote the decomposition space $Y = X/G$.

This example produces an analogue of the metric solenoid in the case that Z is non-metric. We state and prove some of the properties of Y. The fact that the collection G is indeed an upper semi-continuous decomposition X follows from the fact that Y is a compact Hausdorff space, F is a homeomorphism and each element of G is compact.

For ease of notation we will suppress the collection G for the following proofs. Thus, in the case that I is the unit interval with end points $a = 0$ and $b = 1$, we will let $\{\{(F^{-1}(z), b), (z, a)\}, \{(z, b), (F(z), a)\}\} \cup (\{z\} \times (I - \{a, b\}))$ be denoted by $\{z\} \times [0, 1]$ with the identifications of G understood.

The specific resultant space Y will depend on our choices for Z and F. For any Y constructed according to Construction 1 we have the following properties.

Property 1. For each $t \in Z$ the set $\{F^n(t)\}_{n=0}^{\infty}$ is dense in Z.

Proof. Let $t \in Z$ and let O be an open set intersecting Z. Then by hypotheses there is a clopen set S and integer N_S with $z \in S \subset O$ so that $\{F(S)\}_{i=0}^{N_S}$ is a collection of disjoint clopen sets that covers Z. Based on the construction details, note that for the set S required by the definition of the block permutation property we have, $F^{N_S+1}(S) = S$. Then for some integer n, $t \in F^n(S)$. Then since F permutes the elements of $\{F(S)\}_{i=0}^{N_S}$ there is an integer k so that $F^{n+k}(S) = S$. Thus $F^{n+k}(t) \in S \subset O$. □

If F permutes the elements of $\{F(S)\}_{i=0}^{N_S}$ then so does F^{-1} so we have the following:

Corollary 1. *For each $t \in Z$ the set $\{F^{-n}(t)\}_{n=0}^{\infty}$ is dense in Z.*