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Topology of character varieties of Abelian groups ☆

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ABSTRACT

Let G be a complex reductive algebraic group (not necessarily connected), let K be a maximal compact subgroup, and let Γ be a finitely generated Abelian group. We prove that the conjugation orbit space $\mathsf{Hom}(\Gamma, K)/K$ is a strong deformation retract of the GIT quotient space $\mathsf{Hom}(\Gamma, G)/\!\!/G$. Moreover, this result remains true when G is replaced by its locus of real points. As a corollary, we determine necessary and sufficient conditions for the character variety $\mathsf{Hom}(\Gamma, G)/\!\!/G$ to be irreducible when G is connected and semisimple. For a general connected reductive G, analogous conditions are found to be sufficient for irreducibility, when Γ is free Abelian.

1. Introduction

The description of the space of commuting elements in a compact Lie group is an interesting algebro-geometric problem with applications in mathematical physics, remarkably in supersymmetric Yang–Mills theory and mirror symmetry [9,30,51,52]. Some special cases of this problem and related questions have recently received attention, as can be seen, for example, in the articles [1,4,5,18,24,39,46].

Let K be a compact Lie group and view the $\mathbb{Z}$-module $\mathbb{Z}^r$ as a free Abelian group of rank r, for a fixed integer $r > 0$. The space of commuting r-tuples of elements in K can be naturally identified with the set $\mathsf{Hom}(\mathbb{Z}^r, K)$ of group homomorphisms from $\mathbb{Z}^r$ to K, by evaluating a homomorphism on a set of free generators for $\mathbb{Z}^r$. From both representation–theoretic and geometric viewpoints, one is interested in homomorphisms up to conjugacy, so the quotient space $\mathfrak{X}_\Gamma(K) := \mathsf{Hom}(\mathbb{Z}^r, K)/K$, where K acts by

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conjugation, is the main object to consider. Since every compact Lie group is isomorphic to a matrix group, it is not difficult to see that this orbit space is a semialgebraic space, but many of its general properties remain unknown.

In this article, we also consider the analogous space for a complex reductive affine algebraic group G, or its real points $G(\mathbb{R})$. More generally, in many of our results we replace the free Abelian group $\mathbb{Z}^r$ by an arbitrary finitely generated Abelian group Γ.

In this context, it is useful to work with the geometric invariant theory (GIT) quotient space, denoted by $\mathsf{Hom}(\Gamma, G)/\!\!/G$, and usually called the G-character variety of Γ (see Section 2 for details). The character varieties $\mathfrak{X}_\Gamma(G) := \mathsf{Hom}(\Gamma, G)/\!\!/G$ are naturally affine algebraic sets; not necessary irreducible, smooth, nor homotopically trivial. In the case of its real locus $G(\mathbb{R})$, we consider the space of closed orbits, also denoted $\mathfrak{X}_\Gamma(G(\mathbb{R}))$, which is semi-algebraic by [42].

Here is our first main result:

Theorem 1.1. *Let G a complex or real reductive algebraic group, K a maximal compact subgroup of G, and Γ a finitely generated Abelian group. Then there exists a strong deformation retraction from $\mathfrak{X}_\Gamma(G)$ to $\mathfrak{X}_\Gamma(K)$.*

We remark that in [18], the analogous result for complex reductive G was shown for Γ a free (non-Abelian) group F_r of rank r; that is, the free group character variety $\mathsf{Hom}(\mathsf{F}_r, G)/\!\!/G$ deformation retracts to $\mathsf{Hom}(\mathsf{F}_r, K)/K$. In the same article, the special case $K = \mathsf{U}(n)$, $G = \mathsf{GL}(n, \mathbb{C})$ and $\Gamma = \mathbb{Z}^r$ of Theorem 1.1 was also shown. Likewise this result remains true for $G(\mathbb{R})$ and $\Gamma = \mathsf{F}_r$, see [12].

Pettet and Souto in [39] have shown that, under the hypothesis of Theorem 1.1, $\mathsf{Hom}(\mathbb{Z}^r, G)$ deformation retracts to $\mathsf{Hom}(\mathbb{Z}^r, K)$. An intermediate space along their retraction is the space of representations with closed orbits, which is also important for understanding the topology of the GIT quotient. Thus, some steps in their result are useful for our independent proof of Theorem 1.1, which is moreover very different from our argument in [18].

In fact, there are great differences between the free Abelian and free (non-Abelian) group cases. For instance, the deformation retraction from $\mathsf{Hom}(\mathsf{F}_r, G) \cong G^r$ to $\mathsf{Hom}(\mathsf{F}_r, K) \cong K^r$ is trivial, although the deformation from $\mathfrak{X}_{\mathsf{F}_r}(G) = \mathsf{Hom}(\mathsf{F}_r, G)/\!\!/G$ to $\mathfrak{X}_{\mathsf{F}_r}(K) = \mathsf{Hom}(\mathsf{F}_r, K)/K$ is not. Moreover, in the case of free Abelian groups the deformation is not determined in general by the factor-wise deformation (as happens in the free group case), since any such deformation cannot preserve commutativity (see [49]). We conjecture that the analogous result is valid for right angled Artin groups, a class of groups that interpolates between free and free Abelian ones (see Section 4.3).

Returning to the situation of a general finitely generated group Γ, the interest in the spaces $\mathsf{Hom}(\Gamma, K)/K$ and $\mathsf{Hom}(\Gamma, G)/\!\!/G$ is also related to their differential-geometric interpretation. Consider a differentiable manifold B whose fundamental group $\pi_1(B)$ is isomorphic to Γ (when $\Gamma = \mathbb{Z}^r$, we can choose B to be an r-dimensional torus). By fixing a base point in B and using the standard holonomy construction in the differential geometry of principal bundles, one can interpret $\mathsf{Hom}(\Gamma, K)$ as the space of pointed flat connections on principal K-bundles over B, and $\mathfrak{X}_\Gamma(K) = \mathsf{Hom}(\Gamma, K)/K$ as the moduli space of flat connections on principal K-bundles over B (see [51]).

The use of differential and algebro-geometric methods to study the geometry and topology of these spaces was achieved with great success when B is a closed surface of genus $g > 1$ (in this case $\pi_1(B)$ is non-Abelian), via the celebrated Narasimhan–Seshadri theorem and its generalizations, which deal also with non-compact Lie groups (see, for example [3,28,38,47,48]). Indeed, the character varieties $\mathfrak{X}_\Gamma(G)$ introduced above can be interpreted as a moduli space of polystable G-Higgs bundles over a compact Kähler manifold with Γ the fundamental group of the manifold, or central extension thereof (which yields an identification in the topological category, but not in the algebraic or complex analytic ones). Some of the multiple conclusions from this approach was the determination of the number of components for many spaces of the form $\mathsf{Hom}(\pi_1(B), H)/\!\!/H$ for a closed surface B and real reductive (not necessarily compact or complex) Lie