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Locally nonexpansive mappings in geodesic and length spaces

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ABSTRACT

The main focus of this paper is on fixed point theory for mapping satisfying local contractive conditions in Banach spaces and in various geodesic spaces. The emphasis is on locally nonexpansive mappings and local contractions.

1. Introduction

In its broadest sense, metric fixed point theory has been couched in the setting of a complete metric space. When specific structure is needed the setting has usually been in the framework of a Banach space. However there are many intermediate classes of spaces for which there are rich sets of examples and applications. These include geodesic spaces, including the hyperbolic spaces, Busemann convex spaces, and CAT(0), as well as the more general length spaces. For fixed point theory in these spaces, we refer the reader to [1,17,18,26–28]. The main focus of this paper will be on fixed point theory in these spaces for mappings that satisfy local contracting conditions. Local conditions seem naturally suited to the study mappings defined on nonconvex open domains because they may arise from differentiability assumptions (see the discussion in [14]). On the other hand, as we shall see, a major obstacle in proving metric fixed point theorems in length spaces is the difficulty in defining analogs of geometric structure present in certain geodesic spaces. This appears to be a largely underexplored aspect of the theory.

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2. Preliminaries

We begin by fixing some terminology and notation. A *path* in a metric space (X,d) is a continuous image of the unit interval $I=[0,1]\subset\mathbb{R}$. If $S\equiv f(I)$ is a path then its *length* is defined as

$$\ell(S)=\sup_{(x_i)}\sum_{i=0}^{N-1}d\big(f(x_i),f(x_{i+1})\big)$$

where $(x_i)=(0=x_0<x_1<\cdots<x_N=1)$ is any partition of $[0,1]$. If $\ell(S)<\infty$ then the path is said to be *rectifiable*.

A metric space (X,d) which is rectifiably pathwise connected is said to be a *length space* if the distance between each two points x,y of X is the infimum of the lengths of all rectifiable paths joining them. In this case, d is said to be a *length metric* (otherwise known an *inner metric* or *intrinsic metric*).

A length space (X,d) is called a *geodesic space* if there is a path S joining each two points $x,y\in X$ for which $\ell(S)=d(x,y)$. Such a path is often called a *metric segment* (or *geodesic segment*) with endpoints x and y.

Remark. It is important to note that not every pathwise connected metric space is rectifiably pathwise connected. It is enough to consider any normed linear space (or even the real line) with the modified metric defined as the square root of the norm metric. In such spaces any path joining distinct points has infinite length.

Many of our results require the following additional structure.

Definition 1. ([21]) (X,d,W) is called a *hyperbolic space* if (X,d) is a metric space and $W:X\times X\times[0,1]\to X$ is a function satisfying

(i) $\forall x,y,z\in X$ and $\forall\lambda\in[0,1]$, $d(z,W(x,y,\lambda))\le\lambda d(z,x)+(1-\lambda)d(z,y)$;
(ii) $\forall x,y\in X$ and $\forall\lambda_1,\lambda_2\in[0,1]$, $d(W(x,y,\lambda_1),W(x,y,\lambda_2))=|\lambda_1-\lambda_2|d(x,y)$;
(iii) $\forall x,y\in X$ and $\forall\lambda\in[0,1]$, $W(x,y,\lambda)=W(y,x,1-\lambda)$;
(iv) $\forall x,y,z,w\in X$ and $\forall\lambda\in[0,1]$, $d(W(x,z,\lambda),W(y,w,\lambda))\le(1-\lambda)d(x,y)+\lambda d(z,w)$.

We shall adopt the convention of using $\lambda x\oplus(1-\lambda)y$ to denote $W(x,y,\lambda)$. With this notation (i), (ii), and (iv) become, respectively:

(i) $d(z,\lambda x\oplus(1-\lambda)y)\le\lambda d(z,x)+(1-\lambda)d(z,y)$;
(ii) $d(\lambda_1x\oplus(1-\lambda_1)y,\lambda_2x\oplus(1-\lambda_2)y)=|\lambda_1-\lambda_2|d(x,y)$;
(iv) $d(\lambda x\oplus(1-\lambda)z,\lambda y\oplus(1-\lambda)w)\le(1-\lambda)d(x,y)+\lambda d(z,w)$. Moreover, if $z=w$, (iv) reduces to

$$d\big(\lambda x\oplus(1-\lambda)z,\lambda y\oplus(1-\lambda)z\big)\le(1-\lambda)d(x,y).\tag{2.1}$$

As discussed in detail in [21], if only condition (i) is satisfied then (X,d,W) is a convex metric space in the sense of *Takahashi* [29]. The first three conditions are equivalent to saying (X,d,W) is a space of *hyperbolic type* in the sense of [8]. The set

$$[x,y]:=\big\{W(x,y,\lambda):\lambda\in[0,1]\big\}$$

is called the *metric segment* joining x and y. (Condition (iii) ensures that $[x,y]$ is an isometric image of the real line interval $[0,d(x,y)]$.) Hyperbolic spaces include all normed linear spaces and convex subsets thereof,