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Metrization conditions for topological vector spaces with Baire type properties [☆]

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ABSTRACT

We show: (i) A Baire topological vector space is metrizable if and only if it has countable cs^*-character. (ii) A locally convex b-Baire-like space is metrizable if and only if it has countable cs^*-character. Both results extend earlier metrization theorems involving the concept of the cs^*-countable character. Theorem (ii) extends a theorem (Sakai) stating that the space $C_p(X)$ has countable cs^*-character if and only if X is countable.

1. Introduction

All topological spaces and groups are assumed to be Hausdorff. Good sufficient conditions for topological spaces and groups to be metrizable provide a significant problem. The classical metrization theorem of Birkhoff and Kakutani states that a topological group G is metrizable if and only if G is first-countable, i.e. G has a countable open base at the unit e of G.

Banakh and Zdomskyy [1] provided another sufficient condition for a Fréchet-Urysohn topological group to be metrizable. Let x be a point of a topological space X. A countable family $\mathcal{D}_x$ of subsets of X containing x is called a *countable cs^*-network at x* if for each sequence in X converging to x and each neighborhood

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U of x, there is $D \in \mathcal{D}_x$ such that $D \subseteq U$ and D contains infinitely many elements of that sequence. Following [1], a space X has *countable* cs^**-character* if it has a countable cs^*-network at each point $x \in X$. Recall two results of Banakh and Zdomskyy.

Theorem 1.1. *([1]) (i) A topological group G is metrizable if and only if G is Fréchet–Urysohn and has countable cs^*-character. (ii) A Baire topological group G is metrizable if and only if G is sequential and has countable cs^*-character.*

Sakai [7] showed that the Fréchet–Urysohness in Theorem 1.1 can be replaced by an essentially weaker condition. Following Arhangel'skii, a topological space X is said to be *κ-Fréchet–Urysohn* if for every open subset U of X and every $x \in \overline{U}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq U$ converging to x. Note that the class of κ-Fréchet–Urysohn spaces is much wider than the class of Fréchet–Urysohn spaces [4]. Next follows immediately from [7, Lemma 2.14].

Theorem 1.2. *([7]) A topological group G is metrizable if and only if it is κ-Fréchet–Urysohn with countable cs^*-character.*

It turns out that, if the space $C_p(X)$ has countable cs^*-character, then $C_p(X)$ is already a κ-Fréchet–Urysohn space by [7, Lemma 2.11]. Making use of this fact and previous Theorem 1.2, Sakai proved the next remarkable result.

Theorem 1.3. *([7]) The space $C_p(X)$ is metrizable if and only if it has countable cs^*-character.*

Let us mention that the picture for the space $C_c(X)$ of all continuous real-valued functions on X with the compact-open topology is completely different as shown in [9] (see also [2]): for any Polish non-locally compact space X, the space $C_c(X)$ has countable cs^*-character and it is not metrizable.

These three theorems provide a nice application as well as the importance of the concept of countable cs^*-character. Our Theorem 1.4 characterizes countable cs^*-character at any point of a topological space.

We need a few technical notations. Let the product $\mathbb{N}^\mathbb{N}$ be endowed with the natural partial order, i.e., $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $i \in \mathbb{N}$, where $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\beta = (\beta_i)_{i\in\mathbb{N}}$. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$, set $I_k(\alpha) := \{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \text{ for } i = 1, \dots, k\}$.

Theorem 1.4. *Let x be a point of a topological space X. Then X has countable cs^*-character at x if and only if there is a family $\{A_\alpha : \alpha \in \mathbf{M}_x\}$ of subsets of X containing x such that*

- (i) *$\mathbf{M}_x$ is a subset of the partially ordered set $\mathbb{N}^\mathbb{N}$.*
- (ii) *$A_\beta \subseteq A_\alpha$, whenever $\alpha \le \beta$ for $\alpha, \beta \in \mathbf{M}_x$.*
- (iii) *For any neighborhood W at x there is $\alpha \in \mathbf{M}_x$ such that $A_\alpha \subseteq W$.*
- (iv) *For each $\alpha \in \mathbf{M}_x$ and each sequence S converging to x, there is $k \in \mathbb{N}$ such that $D_k(\alpha) \cap S$ is infinite, where $D_k(\alpha) := \bigcap_{\beta \in I_k(\alpha)\cap\mathbf{M}_x} A_\beta$.*

In this case the countable family $\mathcal{D} := \{D_k(\alpha) : \alpha \in \mathbf{M}_x \text{ and } k \in \mathbb{N}\}$ is a countable cs^*-network at x.

Arguably the most important class of topological groups is the class of topological vector spaces (tvs for short). Thus metrization theorem which supplements already known results might be of great use. Sakai [6, Proposition 2.5] proved that if $C_c(X)$ is Baire, then it is κ-Fréchet–Urysohn. This combined with Theorem 1.1(i) implies that every Baire space $C_c(X)$ with countable cs^*-character is metrizable. Our next result extends this theorem to any Baire tvs. Also Theorem 1.4 allows to extend Theorem 1.1(ii) in the realm of tvs by removing the assumption G is sequential.