Topology and its Applications 173 (2014) 175–187

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Fixed subgroups of automorphisms of hyperbolic 3-manifold groups

Jianfeng Lin, Shicheng Wang *

Department of Mathematics, Peking University, Beijing, China

ARTICLE INFO

Article history:
Received 18 February 2014
Accepted 6 May 2014
Available online 4 June 2014

Keywords:
Fixed subgroups
Automorphisms
Hyperbolic 3-manifold

ABSTRACT

For fixed subgroups $Fix(\phi)$ of automorphisms ϕ of hyperbolic 3-manifold groups $\pi_1(M)$, we observe that $\mathrm{rk}(Fix(\phi)) < 2\,\mathrm{rk}(\pi_1(M))$ and the constant 2 in the inequality is sharp; we also classify all possible groups $Fix(\phi)$.

1. Introduction

For a group G and an automorphism $\phi : G \to G$, we define $Fix(\phi) = \{\omega \in G \mid \phi(\omega) = \omega\}$, which is a subgroup of G, and use $\mathrm{rk}(G)$ to denote the rank of G.

The so called Scott conjecture proved 20 years ago in a celebrate work of M. Bestvina and M. Handel [1] states that:

Theorem 1.1. *For each automorphism ϕ of a free group $G = F_n$,*

$$\mathrm{rk}\big(Fix(\phi)\big) \le \mathrm{rk}(G).$$

In a recent paper by B.J. Jiang, S.D. Wang and Q. Zhang [4], it is proved that

Theorem 1.2. *For each automorphisms ϕ of a compact surface group $G = \pi_1(S)$,*

$$\mathrm{rk}\big(Fix(\phi)\big) \le \mathrm{rk}(G).$$

It is obvious that the bounds given in Theorem 1.1 and Theorem 1.2 are sharp and can be achieved by the identity maps.

* Corresponding author.
E-mail addresses: linjian5477@pku.edu.cn (J. Lin), wangsc@math.pku.edu.cn (S. Wang).

http://dx.doi.org/10.1016/j.topol.2014.05.020
0166-8641/

In this note, we will address the similar problem for hyperbolic 3-manifold groups. We call a compact 3-manifold M *hyperbolic*, if M is orientable, and the interior of M admits a complete hyperbolic structure of finite volume (then M is either closed or ∂M is a union of tori). In this case $G = \pi_1(M)$ is isomorphic to a cofinite volume torsion free Kleinian group. A main observation in this paper is the following:

Theorem 1.3. *For each automorphism ϕ of a hyperbolic 3-manifold group, $G = \pi_1(M)$,*

$$\mathrm{rk}\big(Fix(\phi)\big) < 2\,\mathrm{rk}(G),$$

and the upper bound is sharp when G runs over all hyperbolic 3-manifold groups.

Theorem 1.3 is a conclusion of the following Theorem 1.4 and Theorem 1.5.

Theorem 1.4. *There exists a sequence of automorphisms $\phi_n : \pi_1(M_n) \to \pi_1(M_n)$ of closed hyperbolic 3-manifolds M_n such that $Fix(\phi_n)$ is the group of a closed surface, and*

$$\frac{\mathrm{rk}(Fix(\phi_n))}{\mathrm{rk}(\pi_1(M_n))} > 2 - \epsilon \quad \text{as } n \to \infty$$

for any $\epsilon > 0$.

Theorem 1.5. *Suppose ϕ is an automorphism of $G = \pi_1(M)$, where M is a hyperbolic 3-manifold. Then $\mathrm{rk}(Fix(\phi)) < 2\,\mathrm{rk}(G)$.*

The proof of Theorem 1.4 is self-contained up to some primary (and elegant) facts on hyperbolic geometry and on combinatoric topology and group theory. Roughly speaking each (M_i, ϕ_i) in Theorem 1.4 is constructed as follows: We first construct the hyperbolic 3-manifold P_i with connected totally geodesic boundary. Then we double two copies of P_i along their boundaries to get the closed hyperbolic 3-manifold M_i. The reflection of M_i alone ∂P_i induces an automorphism $\phi_i : \pi_1(M_i) \to \pi_1(M_i)$ with $Fix(\phi) = \pi_1(\partial P_i)$. In this process all involved ranks are carefully controlled, we get the inequality in Theorem 1.4.

To prove Theorem 1.5, besides some combinatorial arguments on topology and on group theory, we need the following Theorem 1.6 which classifies all possible groups $Fix(\phi)$ for automorphisms ϕ of hyperbolic 3-manifold groups. Recall that each automorphism ϕ of $\pi_1(M)$ can be realized by an isometry f on M according to Mostow rigidity theorem.

Theorem 1.6. *Suppose $G = \pi_1(M)$, where M is a hyperbolic 3-manifold, and ϕ is an automorphism of G. Then $Fix(\phi)$ is one of the following types: the whole group G; the trivial group $\{e\}$; $\mathbb{Z}$; $\mathbb{Z} \oplus \mathbb{Z}$; a surfaces group $\pi_1(S)$, where S can be orientable or not, and closed or not. More precisely*

(1) *Suppose ϕ is induced by an orientation preserving isometry.*
  (i) *$Fix(\phi)$ is either $\mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z}$, or G, or $\{e\}$; moreover*
  (ii) *If M is closed, then $Fix(\phi)$ is either $\mathbb{Z}$ or G;*
(2) *Suppose ϕ is induced by an orientation reversing isometry f.*
  (i) *If $\phi^2 \neq id$, then $Fix(\phi)$ is either $\mathbb{Z}$ or $\{e\}$;*
  (ii) *If $\phi^2 = id$, then $Fix(\phi)$ is either $\{e\}$, or the surface group $\pi_1(S)$, where S is an embedded surface in M that is pointwise fixed by f.*

Theorem 1.6 is proved by using the algebraic version of Mostow Rigidity theorem, as well as some hyperbolic geometry and covering space argument.