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On the uniqueness of decomposition into Cartesian product of curves

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ABSTRACT

We prove that if $X = X_1 \times X_2 \times \ldots \times X_n$, where X_i is a locally connected curve then X has a unique decomposition into Cartesian product of curves.

1. Introduction

A space X is called prime if it is not homeomorphic to Cartesian product of two spaces, each containing at least two points. A decomposition into Cartesian product of prime factors is in general not unique. For example $[0, 1) \times [0, 1]$ is homeomorphic to $[0, 1) \times [0, 1)$ but the factors are not homeomorphic. In case of infinite product Anderson in [1] proved that infinite product of triods is homeomorphic to the Hilbert cube. In [3] Borsuk asked a question:

Problem 1. Is it true that if continuum X has a decomposition into Cartesian product of curves, then this decomposition is unique?

Although the problem was stated over 60 years ago, we still don't know the answer in general case. There are several partial results. In [2] Borsuk proved that the decomposition of polyhedron into Cartesian product of 1-dimensional factors is unique. In [10] Patkowska gave positive answer to Borsuk's question if

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the factors are ANRs. Furdzik in [8] proved that the product of two locally connected curves has a unique decomposition. Cauty solved in [5] the problem for the product of locally connected curves such that every neighbourhood of every point of curves contains a simple closed curve.

In our paper we give a positive answer to Borsuk's question in case of locally connected curves:

Theorem 1. *If a continuum X can be decomposed into Cartesian product of n locally connected curves then this decomposition is unique.*

In our proof we use earlier results of Patkowska [10], Furdzik [8] and Cauty [5]. The proof is based on the concept of homotopic zones of points similar in spirit to homotopically labil and stabil points introduced in [9] and used by Furdzik in [8]. We use also some Patkowska's tools from [10].

2. Notation and elementary properties

Our terminology follows [6] and [7]. All spaces are assumed to be metric. By dimension we understand the covering dimension. All maps in this paper are continuous.

A homeomorphic image of the set $\{x \in \mathbb{E}^n : |x| \le 1\}$ is called n-dimensional cell and denoted by Q^n. By $\alpha(X)$ we denote points of X having a neighbourhood homeomorphic to $\mathbb{E}^n$, by $\beta(X)$ — points of $X \setminus \alpha(X)$ having a neighbourhood homeomorphic to half Euclidean space and $\gamma(X) = X \setminus (\alpha(X) \cup \beta(X))$.

A continuous mapping $h: X \times I \to X$ is a homotopic deformation if $h(x,0) = x$ for every $x \in X$. Two points x_1 and x_2 are homotopic in X if there exists two homotopic deformations: $h_1, h_2: X \times I \to X$ such that $h_1(x_1, 1) = x_2$ and $h_2(x_2, 1) = x_1$.

Definition 1. The set of points of space X homotopic to the point x is called the *homotopic zone of* x.

Definition 2. Let X be a curve. By $\mu(X)$ we denote the set of all points of X having a neighbourhood being a dendrite. Let $\lambda(X) = X \setminus \mu(X)$.

We begin with elementary results on homotopic zones. The proofs are standard and left for the reader. Some of them can be found in [8,3] and [4]. We give Proposition 8 with a short proof.

Proposition 1. *If $x \in X$ and $h: X \to Y$ is a homeomorphism then the image of the homotopic zone of point x is a homotopic zone of point $h(x)$.*

Proposition 2. *If $X = X_1 \times X_2 \times \ldots \times X_n$ then every homotopic zone in X is the Cartesian product of some homotopic zones of $X_1, X_2, \ldots, X_n$.*

Proposition 3. *If C is a locally connected curve and $x_0 \in \mu(C)$, then for every point $x \in C$ there exists a homotopic deformation $h: C \times I \to C$ such that $h(x_0, 1) = x$.*

Proposition 4. *If C is a locally connected curve and $x_0 \in \lambda(C)$ then for every homotopic deformation $h: C \times I \to C$ we have $h(x_0, 1) = x_0$.*

Proposition 5. *A locally connected curve C has as homotopic zones:*

- *the individual points of $\lambda(C)$,*
- *the set $\mu(C)$.*

Proposition 6. *If $X = X_1 \times X_2 \times \ldots \times X_n$, where X_i is a locally connected curve then X has as homotopic zones the sets of form $C_1 \times C_2 \times \ldots \times C_n$, where C_i is individual point of $\lambda(X_i)$ or $C_i = \mu(X_i)$.*