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Polynomial birack modules

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ABSTRACT

Birack modules are modules over an algebra $\mathbb{Z}[X]$ associated to a finite birack X. In previous work, birack module structures on $\mathbb{Z}_n$ were used to enhance the birack counting invariant. In this paper, we use birack modules over Laurent polynomial rings $\mathbb{Z}_n[q^{\pm 1}]$ to enhance the birack counting invariant, defining a customized Alexander polynomial-style signature for each X-labeled diagram; the multiset of these polynomials is an enhancement of the birack counting invariant. We provide examples to demonstrate that the new invariant is stronger than the unenhanced birack counting invariant and is not determined by the generalized Alexander polynomial.

1. Introduction

Biracks are algebraic structures with axioms motivated by framed oriented Reidemeister moves. They were introduced in [6] in order to define invariants of framed oriented knots and links. In [11] a property of finite biracks called *birack rank* was used to define a computable integer-valued invariant of unframed oriented knots and links called the *integral birack counting invariant*, $\Phi_X^{\mathbb{Z}}$. In [1] an algebra $\mathbb{Z}[X]$ called the *rack algebra* was associated to a finite rack X (a particular type of birack), and modules over $\mathbb{Z}[X]$ were studied in [4]. In [7] rack module structures over $\mathbb{Z}_n$ were used to enhance the rack counting invariant, defining a new invariant Φ_X^M which specializes to the integral rack counting invariant $\Phi_X^{\mathbb{Z}}$ but is generally stronger. In [2] rack modules were generalized to the case of biracks, and birack modules over $\mathbb{Z}_n$ were employed to define enhancements of the birack counting invariant.

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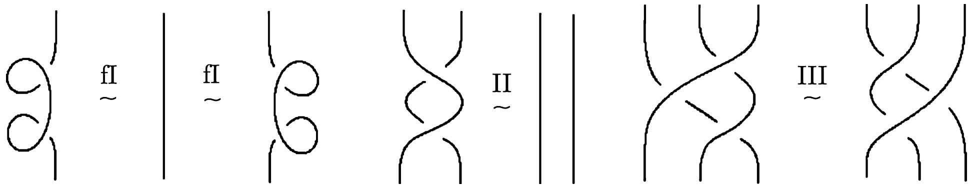
In this paper we consider birack modules over Laurent polynomial rings $\mathbb{Z}_n[q^{\pm 1}]$; such a birack module lets us define a customized Alexander polynomial for each birack homomorphism $f : BR(L) \to X$, the multiset of which forms a new enhancement of the integral counting invariant. Moreover, the generalized Alexander (Sawollek) and classical Alexander polynomials emerge as special cases of the enhanced invariant.

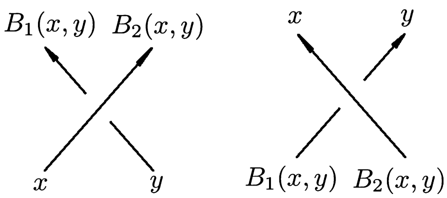
The paper is organized as follows. In Section 2 we review the basics of biracks and birack modules. In Section 3 we define the enhanced link invariant. In Section 4 we collect some examples and in Section 5 we end with some questions for future work.

2. Biracks and birack modules

We begin with a definition. First introduced in [6], a *birack* is an algebraic structure consisting of a set X and a map $B : X \times X \to X \times X$ with axioms derived from the *oriented framed Reidemeister moves*, obtained by considering all ways of orienting the strands in the moves



with the correspondence (also known as the *semiarc labeling rule*)



$$B(x,y) = \big(B_1(x,y), B_2(x,y)\big)$$

In [9] and later in [5] the unframed oriented case, known as the *strong biquandle* case, was considered; the version below comes from [11].

Definition 1. Let X be a set and $\Delta : X \to X \times X$ the diagonal map defined by $\Delta(x) = (x, x)$. Then an invertible map $B : X \times X \to X \times X$ is a *birack map* if the following conditions are satisfied:

(i) There exists a unique invertible map $S : X \times X \to X \times X$ called the *sideways map* such that for all $x, y \in X$, we have

$$S\big(B_1(x,y), x\big) = \big(B_2(x,y), y\big).$$

(ii) The components $(S^{\pm 1}\Delta)_{1,2} : X \to X$ of the composition of the sideways map and its inverse with the diagonal map Δ are bijections, and

(iii) B satisfies the *set-theoretic Yang–Baxter equation*

$$(B \times I)(I \times B)(B \times I) = (I \times B)(B \times I)(I \times B).$$

These axioms are the conditions required to ensure that each labeling of an oriented framed knot or link diagram according to the semiarc labeling rule above before a framed oriented Reidemeister move corresponds to a unique such labeling after the move. By construction, the number of labelings of an