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Bornological convergences and local proximity spaces

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In this paper we clarify the intensive interaction among uniformity, proximity and bornology in local proximity spaces bringing up their underlying uniform characters. By using uniformity and bornology, we give a procedure as an exhaustive method to generate local proximity spaces. We show that the hyperspace of all non-empty closed subsets of a local proximity space carries a very appropriate Fell-type topology, which admits a formulation as hit and far-miss topology and also characterizes as the topology of a two-sided uniform bornological convergence. Furthermore, that topology is induced by the weak uniformity generated by infimum value functionals of the real functions which preserve proximity and boundedness and have a bounded support. Finally, we give necessary and sufficient conditions for topologies of two-sided uniform bornological convergences agree. In particular, we focus on the class of uniform spaces carrying a linearly ordered base, known also as ω_μ-metric spaces. Equipping the hyperspace of an ω_μ-metric space with the Attouch–Wets or bounded Hausdorff topology in the usual way, we achieve among others in the ω_μ-metric setting the following two issues. The former: the Attouch–Wets topologies associated with two ω_μ-metrics on a same space coincide if and only if they have the same bounded sets and are uniformly equivalent on any bounded set. The latter: the Attouch–Wets topologies associated with two ω_μ-metrics on a same space coincide if and only if their hit and bounded far-miss topologies agree.

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1. Introduction

A new reading of the necessary and sufficient conditions for two Attouch–Wets hypertopologies agree [2] and their generalizations [5], which brings up their proximal nature, is the starting point for this paper. Indeed, this strongly suggests the need to connect local proximity spaces with uniform bornological conver-

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gences. A local proximity space [14] is a set equipped with a general proximity [17] and a boundedness [13], i.e., a *proximal universe* by following the primitive definition given by Hu in [13] as assumed in [4–8,12,20], but more, in which the boundedness is a bornology and a *local* family on each member of which the proximity is Efremovič. Local proximity spaces define proximities which, even though not Efremovič, are *locally* Efremovič. The prototype of local proximity space is any T_2 locally compact space carrying as bornology the family of all relatively compact subsets and, in particular, the Euclidean line with its bounded sets. In the T_2 locally compact case, whatever is the compatible uniformity, the uniform bornological convergence on the hyperspace of all closed non-empty subsets with respect to the bornology of relatively compact sets agrees with the natural convergence associated with the Fell topology and, at the same time, as known, is the Kuratowski–Painlevé convergence [3]. In the same vein of a result of Beer [6] showing that a bornological universe satisfying certain conditions can be embedded into a product of real lines equipped with the product topology and a natural product bornology, we embed the underlying space X of a local proximity space in the T_2 local compactification, $\ell(X)$, naturally associated with it [14]. So we make any case to fall within a locally compact one. Indeed, we interpret the topology of certain uniform bornological convergences as a subspace topology of an appropriate Fell topology. Specifically, we show that uniform bornological convergence of nets of closed sets with respect to the Alexandroff uniformity induced on X by $\ell(X)$ is the Kuratowski–Painlevé convergence of an auxiliary net of compact sets in $\ell(X)$.

Let X be a Tychonoff space. Any given T_2-local compactification $\ell(X)$ of X takes up two features of X. The former one is the separated EF-proximity on X induced by the one-point compactification of $\ell(X)$. The latter one is the boundedness done by all subsets of X whose closures in $\ell(X)$ are compact [14,13]. By joining proximity with boundedness in the unique concept of *local proximity space,* S. Leader [14,15] put this example abstractly. Indeed, Leader, paralleling the Smirnov compactification procedure, constructed a bijection between the sets, namely, of all T_2 locally compact dense extensions of X up to equivalence and of all separated local proximities on X. Recently, local proximity spaces have been used very largely in different fields of applications [17,18]. Furthermore, their homeomorphism groups admit as a nice group topology the compact-open topology under weak conditions [10]. In local proximity spaces, uniformity, proximity and bornology coexist and share an intensive interaction. We will try to clarify this. By the means of Weil uniformities [11,21] and bornologies, we display a uniform procedure as an exhaustive method of generating local proximity spaces. Bringing up the underlying uniform characters, we connect local proximity spaces to bornological convergences. In contrast with the proximity case, in which there is no canonical way of equipping the hyperspaces with a uniformity, the same with a proximity, the local proximity case is simpler.

Let $(X, \delta, \mathscr{B})$ be a local proximity space. Apparently, in the beginning, we have two natural different ways to topologize the hyperspace $CL(X)$ of all closed non-empty subsets of X. A first option calls upon the dense embedding of X in the natural T_2 local compactification $\ell(X)$, while a second one stems from joining together proximity and bornology in a hit and far-miss topology. We will show they match in just one case. In a first choice, we identify the hyperspace $CL(X)$ of X with the subspace $\{Cl_{\ell(X)}A : A \in CL(X)\}$ of the hyperspace $CL(\ell(X))$ of $\ell(X)$ when carrying the Fell topology. Then, coming up as a natural mixture, we recast the hit and far-miss topology associated with the proximity δ and the bornology $\mathscr{B}$ as the topology induced by the weak uniformity generated by infimal value functionals of the real functions on X which preserve proximity and boundedness and, moreover, have a bounded support. And finally, we give a further formulation of those as a two-sided uniform bornological convergence. In the seminal paper [16] Lechicki, Levi and Spakowski studied a family of uniform bornological convergences in the hyperspace of a metric space, including the Attouch–Wets, Fell, and Hausdorff metric topologies. We prove the uniform counterpart of the metric case by essentially miming the proofs performed in that by using essentially only uniform features of a metric [5]. In the light of the previous local proximity results, we look for necessary and sufficient conditions of uniform nature for two different uniform bornological convergences to match. Since the metric case is essentially based on two facts, the former: any metrizable uniformity is the finest one in its proximity class, or, in other words, is total; the latter: the bornology of metrically bounded sets

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