Topology and its Applications 173 (2014) 308–324

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Homotopical rigidity of polygonal billiards

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ARTICLE INFO

Article history:
Received 19 April 2013
Received in revised form 5 June 2014
Accepted 5 June 2014
Available online 26 June 2014

Keywords:
Polygonal billiard
Rigidity
Fundamental group

ABSTRACT

Consider two k-gons P and Q. We say that the billiard flows in P and Q are homotopically equivalent if the set of conjugacy classes in the fundamental group of P, viewed as a punctured sphere, which contain a periodic billiard orbit agrees with the analogous set for Q. We study this equivalence relationship and compare it to the notions of order equivalence and code equivalence, introduced in [1,2]. In particular we show if P is a rational polygon, and Q is homotopically equivalent to P, then P and Q are similar, or affinely similar if all sides of P are vertical and horizontal.

1. Introduction

In mathematics one often wants to know if one can reconstruct an object (often a geometric object) from certain discrete data, i.e., does rigidity hold. A well known example is a question posed by Burns and Katok whether a negatively curved surface is determined by its marked length spectrum [4]. The "marked length spectrum" of a surface S is the function that associates to each conjugacy class in the fundamental group $\pi_1(S)$ the length of the geodesic in the class. This question was resolved positively by Otal [10] and Croke [5]. See [11] for a nice survey of rigidity theory.

In this article we consider a related question for polygonal billiards (see [8] for details about polygonal billiards). Precise definitions of all the notions will be given later. We show that the **domain** of the marked length spectrum determines a rational polygon. To understand this statement we need to introduce some notation. Fix P a simply connected polygon with k-vertices. The billiard in P is described by a point mass which moves with unit speed without friction in the interior of P and reflects elastically from the boundary. Remove the vertices and double P, pointwise identifying the two copies of the boundary. This makes P into

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http://dx.doi.org/10.1016/j.topol.2014.06.003

a punctured sphere with k-punctures on the equator E (consisting of the sides of P with the punctures) and a flat metric away from these punctures. We fix one of the two copies of P and call it the *top* hemisphere, the other hemisphere is called the *bottom* hemisphere. This is often called the pillow case model, we will denote the punctured sphere by $\mathbb{P}$ or more precisely by $(\mathbb{P}, E, \{p_1, p_2, \ldots, p_k\})$ where p_i denote the removed points and the flat metric is implicit. We will used the term "arc-side" to denote a side of P in the pillowcase model.

There is a naturally defined 2 to 1 projection map between tangent spaces $T\mathbb{P} \to TP$; considering a billiard trajectory in TP we refer to either of its preimages as a *complete lifting* and also call it a billiard trajectory (in the pillow case model). When a complete lifting arrives to the equator it continues on the other hemisphere of $\mathbb{P}$. The billiard flow $\{T_t\}_{t\in\mathbb{R}}$ in P thus corresponds to the geodesic flow on the unit tangent bundle $T\mathbb{P}$ of this sphere. Denote $\mathrm{proj}_1 : T\mathbb{P} \to \mathbb{P}$ the first natural projection (to the foot point). Let $\hat{\gamma}$ be a complete lifting of a closed billiard trajectory γ in P. Then $\tilde{\gamma} = \mathrm{proj}_1(\hat{\gamma})$ is a closed curve in $\mathbb{P}$ and we will call it a lifting of γ.

Let $D(P)$ be the set of conjugacy classes in $\pi_1(\mathbb{P})$ which contain a lifting of a closed billiard trajectory in P, the set $D(P)$ is the domain of the marked length spectrum map. Consider a polygon Q with the same number of sides as P and call Q homotopically equivalent to P, denoted $Q \equiv_{\mathrm{hom}} P$, if $D(Q) = D(P)$, i.e. each lifting of a closed periodic trajectory in P is homotopic to a lifting of a closed periodic trajectory in Q and vice versa.

In this article we show that if Q is a polygon homotopically equivalent to a rational polygon P (all angles between sides are rational multiples of π) then Q is similar to P (or affinely similar if all sides of P are vertical or horizontal) (Corollary 5.5). Thus among the rational billiard metrics on the k punctured sphere the **domain** of the marked length spectrum determines the polygon up to similarity (resp. affine similarity).

We study the set of free homotopy classes in the k punctured sphere. Motivated by the billiard case we assign symbolic codes to each free homotopy class and show that these codes are in bijection with the free homotopy classes (Theorem 2.6, Corollary 3.1, Proposition 3.9). In Theorem 5.4 we compare the equivalence relation defined above with the notions of order equivalence and code equivalence defined in [1] and [2], as well as a new notion weak code equivalence; we show that under rather natural conditions all of these equivalence relations are the same. Our main result, the aforementioned Corollary 5.5, follows immediately from this theorem. The proof of Theorem 5.4 is based on the bijection between homotopy classes and codes and uses the techniques and results of [2] and [1].

There is a well known construction call the Katok–Zemlyakov construction which for each rational polygon yields a flat surface. In the framework of flat surfaces Duchin et al. have proven a result which at first glance resembles ours. They fix a finite type surface S, i.e. a closed surface with a finite set of marked points removed; then they compare all the flat metrics on S and show that the set of its cylinder curves determine the Teichmüller disc of the flat metric, i.e. the flat metric is determined by its cylinder curves up to affine diffeomorphisms [6, Lemma 22]. This result is close to ours since the set of cylinder curves, as a subset of the fundamental group, is simply the domain of the marked length spectrum function, however the authors do not make this interpretation, rather they prove that if two flat metrics of S have the same marked length spectrum function then they have same set of cylinder curves [6, Corollary 21], and this immediately implies their main theorem, that the marked length spectrum determines the Teichmüller disc of S [6, Theorem 1].

Our result has two significant differences with this result. First we do not fix the genus of our surface, but rather the number of sides of the polygon; in particular the objects we are comparing do not live in a fixed Teichmüller space, we even compare with irrational polygons for which the corresponding flat surface is not even compact. Another difference of our results is the following, for Duchin et al. the fundamental group is that of the surface, while for us the fundamental group is that of the pillowcase. To take a concrete example, if we start with the $(\pi/4, \pi/4, \pi/2)$ triangle, for us the fundamental group is that of the 3-punctured sphere, and we compare it to **all** other triangles. The flat surface corresponding to this triangle is the torus, thus if we restrict the setting of [6] to flat surfaces arising from triangles, we would only be comparing it to triangles