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Topological games and productively countably tight spaces ☆

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ABSTRACT

The two main results of this work are the following: if a space X is such that player II has a winning strategy in the game $\mathsf{G}_1(\Omega_x, \Omega_x)$ for every $x \in X$, then X is productively countably tight. On the other hand, if a space is productively countably tight, then $\mathsf{S}_1(\Omega_x, \Omega_x)$ holds for every $x \in X$. With these results, several other results follow, using some characterizations made by Uspenskii and Scheepers.

1. Introduction

Recall that a topological space X is said to have **countable tightness at a point** $x \in X$ if, for every subset $A \subset X$ such that $x \in \overline{A}$, there is a subset $B \subset A$ such that $x \in \overline{B}$ and B is countable. If X has countable tightness at every point x, then we simply say that X has countable tightness or even that X is countably tight. The tightness does not have a good behavior in products. It is well known that the square of a space of countable tightness may fail to have countable tightness. An internal characterization of those spaces X such that $X \times Y$ has countable tightness for every countably tight Y was given by Arhangel'skii [1]. Although Arhangel'skii's result works for all values of the tightness, here we will focus on the countable case only. Let us say that a topological space X is **productively countably tight** if, for every countably tight space Y, $X \times Y$ has countable tightness. Similarly, X is productively countably tight at a point $x \in X$ provided that, for any space Y which has countable tightness at a point $y \in Y$, the product $X \times Y$ has countable tightness at $\langle x, y\rangle$. In this work, we will show the relation of this productive property with some topological games. Let us introduce the game notation that we will use. Given two families $\mathcal{A}, \mathcal{B}$, we use $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ if, for every sequence $(A_n)_{n\in\omega}$ of elements of $\mathcal{A}$, one can select $a_n \in A_n$ such that $\{a_n : n \in \omega\} \in \mathcal{B}$. Similarly, we use the notation $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ for the game played between players I and II in such a way that,

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for every inning $n \in \omega$, player I chooses a member $A_n \in \mathcal{A}$. Then player II chooses $a_n \in A_n$. Player II is declared the winner if, and only if, $\{a_n : n \in \omega\} \in \mathcal{B}$. For this matter, we will use the following families:

- Ω: the collection of all open ω-coverings for a space (recall that $\mathcal{C}$ is a ω-covering if, for every $F \subset X$ finite, there is a $C \in \mathcal{C}$ such that $F \subset C$);
- Ω_x: the collection of all sets A such that $x \notin A$ and $x \in \overline{A}$.

In Section 2 we will prove that if player II has a winning strategy in the game $\mathsf{G}_1(\Omega_x, \Omega_x)$ for every $x \in X$, then X is productively countably tight. On the other hand, if X is productively countably tight, then X has the property $\mathsf{S}_1(\Omega_x, \Omega_x)$ for every $x \in X$. Recall that $\mathsf{S}_1(\Omega_x, \Omega_x)$ means that X has countable strong fan tightness at x.

In Section 3, we use some translations of the properties used here to the spaces of the form $C_p(X)$. This kind of translation allow us to show some new results, even ones that do not involve spaces of the form $C_p(X)$. Like, per example, if X is a Tychonoff space and player II has a winning strategy in the game $\mathsf{G}_1(\Omega, \Omega)$, then the G_δ modification of X is Lindelöf.

Finally, in Section 4 we present some examples in order to show that the implications made in the previous sections cannot be reversed.

2. Productively countably tight spaces

According to Arhangel'skii [1], a topological space X is $\aleph_0$-**singular** at a point $x \in X$ provided that there exists a collection $\mathcal{P}$ of centered[1] families of countable subsets of X such that

(1) for any neighborhood O_x of x there exist $\mathcal{B} \in \mathcal{P}$ and $B \in \mathcal{B}$ such that $B \subset O_x$;
(2) for any $\{\mathcal{B}_n : n < \omega\} \subset \mathcal{P}$ we may pick $B_n \in \mathcal{B}_n$ in such a way that $x \notin \overline{\bigcup\{B_n : n < \omega\}}$.

Theorem 2.1. *(Arhangel'skii [1, Theorem 3.4]) Given a Tychonoff space X and a point $x \in X$, X is productively countably tight at x if, and only if, X is not $\aleph_0$-singular at x.*

With the help of this characterization, we will prove the following:

Theorem 2.2. *Let X be a space such that player II has a winning strategy F in the game $\mathsf{G}_1(\Omega_x, \Omega_x)$ for some $x \in X$. Let $\mathcal{P}$ be a family such that each element of $\mathcal{P}$ is a centered family of countable subsets of X satisfying (1) in the definition of $\aleph_0$-singularity. Then, there is a family $(\mathcal{B}_s)_{s \in \omega^{<\omega}}$ of elements of $\mathcal{P}$ such that for every choice $B_s \in \mathcal{B}_s$ for each $s \in \omega^{<\omega}$, $x \in \overline{\bigcup_{s \subset f} B_s}$ for any $f \in \omega^\omega$.*

In order to prove this theorem, we will use the following lemma:

Lemma 2.3. *Let X be a space and let F be a strategy for player II in the game $\mathsf{G}_1(\Omega_x, \Omega_x)$ for some $x \in X$. Then, for every sequence $D_0, \ldots, D_n \in \Omega_x$, there is an open set A such that $x \in A$ and, for every $a \in A \setminus \{x\}$, there is a $D_a \in \Omega_x$ such that $F(D_0, \ldots, D_n, D_a) = a$.*

Proof. Let $B = \{y \in X \setminus \{x\}$: there is no $D \in \Omega_x$ such that $F(D_0, \ldots, D_n, D) = y\}$. Note that $B \notin \Omega_x$ since, otherwise, $F(D_0, \ldots, D_n, B) \in B$ which is a contradiction. Thus, there is an open set A such that $x \in A$ and $A \cap B = \emptyset$. □

[1] A family $\mathcal{F}$ is centered if, for every $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.