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On the topology of the inverse limit of a branched covering over a Riemann surface

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ABSTRACT

We introduce the Plaque Topology on the inverse limit of a branched covering self-map of a Riemann surface of a finite degree greater than one. We present the notions of regular and irregular points in the setting of this Plaque Inverse Limit and study its local topological properties at the irregular points. We construct a certain Boolean algebra and a certain sigma-lattice, derived from it, and use them to compute local topological invariants of the Plaque Inverse Limit. Finally, we obtain several results interrelating the dynamics of the forward iterations of the self-map and the topology of the Plaque Inverse Limit.

1. Introduction

Topological inverse limits of dynamical systems were constructed and their topological properties were studied in literature since the late 1920s. The most famous classical examples of such inverse limits are the solenoids, which are defined as the inverse limits of the iterates of the d-fold covering self-mapping $f(z) = z^d$ (where $d > 1$) of the unit circle S^1. The inverse limit of these iterates, for a fixed integer d, is called the d-adic solenoid. It is a compact, metrizable topological space that is connected, but neither locally connected nor path connected. Solenoids were first introduced by L. Vietoris in 1927 for $d = 2$ (see [21]) and later in 1930 by van Dantzig for an arbitrary d (see [8]).

D. Sullivan in [20] introduced Riemann surface laminations, which arise when taking inverse limits in dynamics. A Riemann surface lamination is locally the product of a complex disk and a Cantor set. In particular, D. Sullivan associates such lamination to any smooth, expanding self-mapping of the circle S^1, with the maps $f(z) = z^d$ being examples of such mappings.

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M. Lyubich and Y. Minsky took it one dimension higher. In [14] they consider dynamics of rational self-mappings of the Riemann sphere and introduce three-dimensional laminations associated with these dynamics. Thus, the theory of Riemann surface laminations associated with holomorphic dynamics was first founded and formalized by M. Lyubich and Y. Minsky in [14] and, in parallel, by M. Su in [19]. The notions of regular points and, by complement, irregular points were introduced in [14]. First, M. Lyubich and Y. Minsky consider the standard (Tychonoff) inverse limit of the iterations of a rational map applied to the Riemann sphere, regarding them just as iterations of a continuous branched covering map applied to a Hausdorff topological space. They call this inverse limit the natural extension of the original dynamical system. Next, they define a point of this natural extension to be regular if the pull-back of some open neighborhood of its first coordinate along that point will eventually become univalent. The set of all regular points of the natural extension, which is the natural extension with all the irregular points removed, is called the regular set. The Riemann surface lamination, which in the Lyubich–Minsky theory is associated with a holomorphic dynamical system, in many cases, is just the regular set. In general, certain modifications are performed to the regular set, in order to satisfy the requirement, that the conformal structure on the leaves of the Riemann surface lamination is continuous along the fiber of the lamination. For the details of Lyubich–Minsky's definition and construction of the Riemann surface lamination, which are somewhat elaborate, we refer to [14].

In this paper we consider the following two questions:

- How can the irregular points be distinguished and characterized?
- What is the relationship between the dynamics of the system and the characterization of the irregular points?

For our purposes it is more natural to equip the inverse limit with the box topology, which is more refined than the Tychonoff topology.

An inverse dynamical system is a sequence:

$$S_1 \xleftarrow{f_1} S_2 \xleftarrow{f_2} S_3 \ldots$$

of Riemann surfaces S_i and holomorphic branched coverings $f_i : S_{i+1} \to S_i$ where all S_i are equal to a given Riemann surface S_0 and all f_i are equal to a given holomorphic branched covering map $f : S_0 \to S_0$ of degree d. In this work we assume that $1 < d < \infty$ and S_0 is either the unit disk, the complex plane or the Riemann sphere. We define the Plaque Inverse Limit [P.I.L.] S_∞ of that inverse dynamical system to be the following topological space: its underlying set is the set of all the sequences $x = (x_1 \in S_1, x_2 \in S_2, \ldots)$ of points, such that $f_i(x_{i+1}) = x_i$ for $i = 1, 2, \ldots$; its topology is the family of all the sequences $U = (U_1 \subset S_1, U_2 \subset S_2, \ldots)$ of open sets, such that $f_i(U_{i+1}) = U_i$ for $i = 1, 2, \ldots$. Let p_i, for all i, be the map from S_∞ onto S_i, which takes $(x_1, x_2, \ldots) \in S_\infty$ to $x_i \in S_i$. The maps p_i are continuous and satisfy $f_i \circ p_{i+1} = p_i$. They are called the projection maps from S_∞ onto S_i.

The standard Inverse Limit $\bar{S}_\infty$ of f, as a set, is defined exactly as P.I.L., but is equipped with the Tychonoff topology, in which the open sets are all sequences $U = (U_1 \subset S_1, U_2 \subset S_2, \ldots)$ of open sets, where $f_i(U_{i+1}) = U_i$ for $i = 1, 2, \ldots$, such that there exists some number t, so that $f_i^{-1}(U_i) = U_{i+1}$ for all $i > t$. Thus, P.I.L. has more open sets than the standard Inverse Limit. So, the identity map from the P.I.L. onto the Inverse Limit is continuous.

Obviously, the above mentioned projection maps p_i are also continuous, as maps from $\bar{S}_\infty$ onto S_i. Actually, the categorical definition of the topology of the inverse limit of an inverse system is precisely the "minimal" topology, which makes these projection maps p_i continuous. Minimality, in this context, means that any other topological space, together with maps from it into the inverse system, which commute with the maps of the inverse system, can be mapped into the inverse system through the inverse limit, and this