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Generalization of sequences and convergence in metric spaces

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ABSTRACT

Adding to the previous results by the author and using some generalization of sequences, we study a special case of countable decomposability of functions: representation of functions as open, closed and continuous ones with the possible exception of countably many points.

1. Introduction

As we have shown in [1–3], the functions that take open or closed sets into a combination of n open or closed sets for $n = 2$ are decomposable into countably many closed, open and continuous functions.

In this paper we will be focusing on an important special case of countable decomposability of functions: representation of functions as open, closed or continuous ones with the possible exception of countably many points.

We will use countable compact sets $S_n(y)$ and their subsets as a generalization of sequences in metrizable spaces.

1.1. Countable compact sets $S_n(y)$

We will denote by $S_n(y)$ a standard countable compact subset of a space Y such that its n-th derived set $(S_n(y))^n$ is a singleton of y. We denote by $I_n(y)$ the set of isolated points of $S_n(y)$.

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In particular, $I_1(y) = \{y_k\}_{k=1}^{\infty}$ such that $y_k \to y$ as $k \to \infty$; and

$$S_1(y) = \{y\} \cup \{y_k\}$$

is a standard sequence with its limit point, and $I_2(y) = \{y_{k,n}\}_{n=1}^{\infty}$ such that $y_{k,n} \to y_k$ as $k \to \infty$, and

$$S_2(y) = \{y\} \cup \{y_k\} \cup \{y_{k,n}\}$$

1.2. Binary classes A_n and M_n

Starting from classes A_1 and M_1 of open sets and closed sets lying in a metric space, we define the following additive and multiplicative classes:

$$A_n = \{A \cup B : A \in M_{n-1}, B \in A_{n-1}\}$$

and

$$M_n = \{A \cap B : A \in M_{n-1}, B \in A_{n-1}\}$$

By definition, the elements of classes A_n and M_n, called A_n-sets and M_n-sets, form a class B_n. Obviously, $\emptyset \in B_n$ and $B_n \subset B_{n+1}$.

We say that $f : X \to Y$ is *open-B_n* (resp., *closed-B_n*) iff the image of every open (resp., closed) subset is a B_n-set.

We say that a function $f : X \to Y$ is *B_n-measurable* iff the preimage of every open subset is a B_n-set. In this case f^{-1} is a (multivalued) open-B_n function. Below we will consider only single-valued functions.

Continuous (resp., open, closed) functions are A_1-measurable (resp., open-A_1, closed-M_1). Their countable characteristics in terms of $S_1(y)$ are well known. For example, a function $f : X \to Y$ is continuous iff one of the following equivalent conditions holds:

- for each open set $U \subset Y$, its preimage $f^{-1}(U)$ is an A_1-set in X;
- for each closed set $F \subset Y$, its preimage $f^{-1}(F)$ is an M_1-set in X;
- for each $S_1(x) \subset X$, $f(x) \in cl_Y f(I_1(x))$.

2. s_2-Functions

We will consider a *standard base* $\mathcal{B}$ of a zero-dimensional metric space X: this is any base $\mathcal{B}$ consisting of clopen subsets such that, for any set F and $n \in \mathbb{N}^+$, the family $\{V \in \mathcal{B} : V \cap F \neq \emptyset, \operatorname{diam}(V) > 1/n\}$ is finite.

Obviously, each subspace of the Cantor set $\mathbf{C}$ has a standard base.

In this section, we generalize the notions of closed, open and continuous functions with the use of $S_2(x)$-sets.

2.1. s_2-Continuous functions

A function $f : X \to Y$ is said to be s_n-continuous if, for any $S_n(x) \subset X$, $f(x)$ is a limit point of $f(I_n(x))$. It is obvious that a function is continuous iff it is s_1-continuous.

Theorem 1. *Let $f : X \to Y$ be an s_2-continuous function between $X, Y \subset \mathbf{C}$. Then there exists a countable set $Y_0 \subset Y$ such that the restriction $f|(X \setminus f^{-1}(Y_0))$ is continuous.*