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Pseudo-solenoids are not continuously homogeneous

Frank Sturm [1]

Auburn University, Auburn, AL, United States

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ABSTRACT

The topological space X is a pseudo-solenoid if X is a hereditarily indecomposable, non-chainable, and circle-like continuum. It is shown that no such continuum is continuously homogeneous. Specifically, there are two points in X, identified as z and $\bar{1}$, such that for any continuous surjection $f : X \to X$, $f(\bar{1}) \neq z$. This answers a question of J.J. Charatonik in the negative.

1. Introduction

A *continuum* X is a compact and connected metric space. A continuum is *circle-like* (resp. *arc-like or chainable*) if it is homeomorphic to an inverse limit of circles (resp. arcs). A continuum X is *indecomposable* if X is not the union of two proper subcontinua of X; X is *hereditarily indecomposable*, if each subcontinuum of X is indecomposable. Knaster constructed the first example of a hereditarily indecomposable continuum in 1922 [23]. In 1948 Moise constructed an example of a continua, which was hereditarily indecomposable, to answer a question of Mackiewicz [33]. Later that same year, Bing constructed an example similar to Moise's to answer a question of Knaster and Kuratowski [6]. In 1951, Bing showed that all nondegenerate, chainable, and hereditarily indecomposable continua are equivalent up to homeomorphism [7]. From this, it followed that the examples constructed by Knaster, Moise, and Bing are the same space. This space is called the pseudo-arc.

E-mail address: fhs0001@auburn.edu.

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A continuum is a *pseudo-solenoid* if it is not chainable, circle-like, and hereditarily indecomposable. The *pseudo-circle* refers to the topologically unique pseudo-solenoid embeddable in the plane [16]. It is the first example of a pseudo-solenoid, and it was constructed by Bing as an example of a continuum that is hereditarily indecomposable and plane separating [7]. Pseudo-circles and pseudo-solenoids have been of interest in continuum theory since their inception in the literature. They are relatively simple examples of hereditarily indecomposable continua (for recent results see [4,5,9,10,17]). Handel's work [20] showed the pseudo-circle can result as a strange attractor of a smooth diffeomorphism of the plane. Since then dynamicists have taken interest in the pseudo-circle as an invariant set. Among those, Kennedy and Yorke showed that a diffeomorphism with invariant pseudo-circle need not be pathological [22].

A topological space X is *homogeneous* if for each $x, y \in X$ there is a homeomorphism $h : X \to X$ with $h(x) = y$, or equivalently, X has one orbit under the action of its homeomorphism group. Bing and Moise independently showed that the pseudo-arc is homogeneous [6,34] (for a survey of various homogeneity properties of the pseudo-arc see [29,30]). Bing asked if the pseudo-circle is homogeneous as well [7]. Bing's question was answered independently by Fearnley and Rogers, each of whom demonstrated that the pseudo-circle (and every pseudo-solenoid) is not homogeneous [15,39]. Since then, the same result has been obtained using other methods, first by Hagopian [18], using fixed point properties and a theorem of Effros. Subsequent proofs by other authors include two additional proofs by Rogers [41,42], a proof by Lewis [28], and a proof by Kuperberg and Gammon [26]. An interesting extension of this result is due to Kennedy and Rogers [21], who use a theorem of Hagopian to show that the pseudo-circle has uncountably many orbits under the action of its homeomorphism group. Moreover, they also show that each orbit contains uncountably many composants using a technique of Lewis [27]. Recall that two points of an indecomposable continuum X are in the same *composant* if there is a proper subcontinuum of X that contains both points.

In this paper we will refer to a continuous function as a *map*; a *self-map* of a topological space X is a map with X as both the domain and codomain. In the early 1970s, J.J. Charatonik proposed a generalization of topological homogeneity of a topological space X with respect to a class of self-maps of X. A topological space X is $\mathcal{C}$-*homogeneous*, where $\mathcal{C}$ is a class of self-maps of X, if for any $x, y \in X$, there is a surjection $f \in \mathcal{C}$ such that $f(x) = y$. Thus, X is homogeneous in the conventional sense, if $\mathcal{C}$ is the class of all homeomorphisms of X. If $\mathcal{C}$ is the class of all continuous (resp. open or monotone) maps of X, then X is called *continuously* (resp. *openly or monotonically*) *homogeneous*.

Charatonik showed interest in the classes of open and monotone maps of a space and showed that a nondegenerate arc-like continuum is openly homogeneous if and only if it is the pseudo-arc [11], which generalizes a characterization of Bing [8]. Similarly, Prajs generalized a result of Hagopian [19] by showing that the only class of nondegenerate, openly homogeneous, circle-like continua, with all nondegenerate proper subcontinua being arcs, is in fact the class of solenoids [35]. The question thus posed by Charatonik was whether open-homogeneity is equivalent to homogeneity for continua (Question 1 in [12,13]).

Approximately 15 years later the above question of Charatonik was answered in the negative, with independent results by Prajs and Seaquist [36,37,43,44]. Starting with the Sierpiński universal plane curve, which Krasinkiewicz originally proved as having two orbits under the action of its homeomorphism group [24], Prajs and Seaquist each constructed a continuous decomposition of this continuum. It was observed that for either orbit of the Sierpiński curve, there is a point whose preimage under the quotient map intersects both orbits. Hence, the continuum is openly (and monotonically) homogeneous via a homeomorphism composed with the quotient map. However, before these results, Charatonik proposed the pseudo-circle as a possible counterexample, and he remained interested in the question of whether the pseudo-circle is openly (or continuously) homogeneous even after the aforementioned results of Prajs and Seaquist [12]. In his 2006 (English translation) biographical article on Charatonik, Krupski mentions this problem explicitly (see [25, p. vii]).

The various constructions of the pseudo-circle closely resemble those of the pseudo-arc, and it is easily observed that every proper subcontinuum of a pseudo-circle is a pseudo-arc. Moreover, the surjective self-