Topology and its Applications 163 (2014) 66–76

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Induced mappings between quotient spaces of symmetric products of continua

Enrique Castañeda-Alvarado, Fernando Orozco-Zitli, Javier Sánchez-Martínez *

Universidad Autónoma del Estado de México, Facultad de Ciencias, Instituto Literario No. 100, Col. Centro, C.P. 50000, Toluca, Estado de México, Mexico

ARTICLE INFO

MSC:
primary 54C05, 54C10, 54B20
secondary 54B15

Keywords:
Continuum
Hyperspace
Induced mappings
Quotient space
Symmetric product

ABSTRACT

Given a continuum X and $n \in \mathbb{N}$. Let $\mathcal{H}(X) \in \{2^X, C(X), F_n(X)\}$ be a hyperspace of X, where 2^X, $C(X)$ and $F_n(X)$ are the hyperspaces of all nonempty closed subsets of X, all subcontinua of X and all nonempty subsets of X with at most n points, respectively, with the Hausdorff metric. For a mapping $f : X \to Y$ between continua, let $\mathcal{H}(f) : \mathcal{H}(X) \to \mathcal{H}(Y)$ be the induced mapping by f, given by $\mathcal{H}(f)(A) = f(A)$. On the other hand, for $1 \leqslant m < n$, $SF^n_m(X)$ denotes the quotient space $F_n(X)/F_m(X)$ and similarly, let $SF^n_m(f)$ denote the natural induced mapping between $SF^n_m(X)$ and $SF^n_m(Y)$. In this paper we prove some relationships between the mappings f, 2^f, $C(f)$, $F_n(f)$ and $SF^n_m(f)$ for the following classes of mapping: atomic, confluent, light, monotone, open, OM, weakly confluent, hereditarily weakly confluent.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. Given a continuum X. Denote by 2^X the hyperspace of all nonempty closed subsets of X, by $C(X)$ the hyperspace of subcontinua of X and by $F_n(X)$ the hyperspace of all nonempty subsets of X having at most n points, where n is a positive integer. All hyperspaces are considered with the Hausdorff metric (see [20, p. 1]).

Given two positive integers $n > m$, $SF^n_m(X)$ denotes the quotient space $F_n(X)/F_m(X)$ obtained by shrinking $F_m(X)$ to a point in $F_n(X)$, with the quotient topology. The fact that $SF^n_m(X)$ is a continuum follows from Theorem 3.10 of [21, p. 40]. A *mapping* means a continuous function. The quotient mapping from $F_n(X)$ onto $SF^n_m(X)$ is denoted by $\rho^X_{n,m}$.

Given a mapping $f : X \to Y$ between continua, the mappings $2^f : 2^X \to 2^Y$ given by $2^f(A) = f(A)$, $C(f) : C(X) \to C(Y)$ given by $C(f) = 2^f|_{C(X)}$ and $F_n(f) : F_n(X) \to F_n(Y)$ given by $F_n(f) = 2^f|_{F_n(X)}$ are the induced mappings by f. Also, we have an induced mapping $SF^n_m(f) : SF^n_m(X) \to SF^n_m(Y)$ given by

$$SF^n_m(f)\big(\rho^X_{n,m}(A)\big) = \rho^Y_{n,m}\big(f(A)\big).$$

* Corresponding author.
E-mail addresses: eca@uaemex.mx (E. Castañeda-Alvarado), forozcozitli@gmail.com (F. Orozco-Zitli), jsanchezm@uaemex.mx (J. Sánchez-Martínez).

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.10.007

By [11, Theorem 4.3, p. 126], $SF^n_m(f)$ is a mapping and the following diagram

$$\begin{array}{ccc} F_n(X) & \xrightarrow{F_n(f)} & F_n(Y) \\ {\scriptstyle \rho^X_{n,m}}\downarrow & & \downarrow{\scriptstyle \rho^Y_{n,m}} \\ SF^n_m(Y) & \xrightarrow[SF^n_m(f)]{} & SF^n_m(Y) \end{array}$$

is commutative.

Let $\mathcal{M}$ be a class of mappings between continua. A general problem is to find all possible relationships among the following four statements:

(1) $f \in \mathcal{M}$;
(2) $2^f \in \mathcal{M}$;
(3) $C(f) \in \mathcal{M}$;
(4) $F_n(f) \in \mathcal{M}$.

Readers especially interested in this problem are referred to [1–8,12–17].

We can also consider the condition:

(5) $SF^n_m(f) \in \mathcal{M}$.

In this paper we study the interrelations among the statements (1)–(5), for the following classes of mappings: open, monotone, confluent, weakly confluent, OM, light and atomic.

2. Preliminaries

Let X be a continuum with metric d. Let $\delta > 0$ and $A \subset X$, we define $\mathcal{V}^X_\delta(A) = \{x \in X\colon$ there exists $y \in A$ such that $d(x,y) < \delta\}$, and we use the symbol $\mathrm{Cl}(A)$ to denote the closure of A in X.

Given a finite collection $K_1, \dots, K_r$ of subsets of X, $\langle K_1, \dots, K_r\rangle$, denotes the following subset of $F_n(X)$:

$$\left\{A \in F_n(X)\colon\ A \subset \bigcup_{i=1}^{r} K_i,\ A \cap K_i \neq \emptyset \text{ for each } i \in \{1, \dots, r\}\right\}.$$

It is known that the family of all subsets of $F_n(X)$ of the form $\langle K_1, \dots, K_r\rangle$, where each K_i is an open subset of X, forms a basis for a topology for $F_n(X)$ (see [20, Theorem 0.11, p. 9]) called the *Vietoris Topology*. The Vietoris topology and the topology induced by the Hausdorff metric coincide (see [20, Theorem 0.13, p. 10]). If $K_1, \dots, K_r$ are subcontinua of X and $r \leqslant n$, $\langle K_1, \dots, K_r\rangle$ is a subcontinuum of $F_n(X)$ (see [19, Lemma 1, p. 230]).

Given two positive integers $s > k$, we define $\mathcal{F}^s_k(X) = F_s(X) \setminus F_k(X)$. Notice that $\rho^X_{s,k}|_{\mathcal{F}^s_k(X)} : \mathcal{F}^s_k(X) \to SF^s_k(X) \setminus \rho^X_{s,k}(F_k(X))$ is a homeomorphism.

A mapping $f : X \to Y$ between continua is said to be:

– *atomic* if for each subcontinuum K of X such that $f(K)$ is nondegenerate, $f^{-1}(f(K)) = K$;
– *confluent* if for each subcontinuum K of Y and for each component M of $f^{-1}(K)$, $f(M) = K$;
– *light* if $f^{-1}(y)$ is totally disconnected for each $y \in Y$;
– *monotone* if $f^{-1}(y)$ is connected in X for each $y \in Y$;
– *open* if $f(U)$ is open in Y for each open subset U of X;