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Reflexivity in precompact groups and extensions ☆

J. Galindo [a,*], M. Tkachenko [b], M. Bruguera [c], C. Hernández [b]

[a] IMAC, Instituto Universitario de Matemáticas y Aplicaciones, Universidad Jaume I, E-12071, Castellón, Spain
[b] Depto. de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, C.P. 09340, Mexico D.F., Mexico
[c] Dept. de Matemática Aplicada I, Universidad Politécnica de Cataluña, Barcelona, Spain

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A B S T R A C T

We establish some general principles and find some counter-examples concerning the Pontryagin reflexivity of precompact groups and P-groups. We prove in particular that:

(1) A precompact Abelian group G of bounded order is reflexive iff the dual group $G^\wedge$ has no infinite compact subsets and every compact subset of G is contained in a compact subgroup of G.
(2) Any extension of a reflexive P-group by another reflexive P-group is again reflexive.

We show on the other hand that an extension of a compact group by a reflexive ω-bounded group (even dual to a reflexive P-group) can fail to be reflexive.
We also show that the P-modification of a reflexive σ-compact group can be non-reflexive (even if, as proved in [20], the P-modification of a locally compact Abelian group is always reflexive).

1. Introduction

The papers [1,19–21] have unveiled that the duality properties of the class of precompact groups are more complicated than expected. The following theorem summarizes some of the known facts that concern the duality of precompact groups, see below for unexplained terminology.

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* Corresponding author.
E-mail addresses: jgalindo@mat.uji.es (J. Galindo), mich@xanum.uam.mx (M. Tkachenko), m.montserrat.bruguera@upc.es (M. Bruguera), chg@xanum.uam.mx (C. Hernández).

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Theorem 1.1. *Let G be an Abelian group and let τ_H be a precompact topology on G induced by some group of homomorphisms $H \subset \mathrm{Hom}(G, \mathbb{T})$. The topological group $(G, \tau_H)^\wedge$ dual to the precompact group (G, τ_H) can be*:

(1) *A discrete group, as for instance when H is countable (see [3] and [9]) or (G, τ_H) is the Σ-product of uncountably many copies of the discrete group $\mathbb{Z}(2)$ (this can be deduced from (the proof of) Lemma 27.11 of [5], see Lemma 5.2 below). In this case (G, τ_H) is not reflexive.*
(2) *A nondiscrete P-group. This is the case when (G, τ_H) is the ω-bounded group that arises as the dual of a reflexive P-group, as those constructed in [20] and [21]. Obviously (G, τ_H) is reflexive in this case.*
(3) *A precompact non-compact group, as is the case of the infinite pseudocompact groups with no infinite compact subsets constructed in [19] and [1]. These groups are reflexive.*
(4) *A compact group, as happens when $H = \mathrm{Hom}(G, \mathbb{T})$, the family of all homomorphisms of G to $\mathbb{T}$.*

The bases on which the reflexivity of precompact groups stands remain elusive so far. In this paper we give a first insight to this issue by establishing some general facts and giving some counter-examples to what could be regarded as reasonable generalizations of known results. We prove in Proposition 2.10 that a precompact Abelian group of bounded order is reflexive if and only if the compact subsets of the dual group $G^\wedge$ are finite and every compact subset of G is contained in a compact subgroup.

We will especially address the behavior of reflexivity under extensions in precompact, ω-bounded, and P-groups. We recall that the class of P-groups is naturally linked to that of precompact groups through duality since, by [20, Lemma 4.1], the dual group $G^\wedge$ of every P-group G is ω-bounded and hence precompact.

Our starting point in this regard is the fact that an extension of a reflexive group by a compact group is again reflexive provided that the dual of the extension separates points of the extension (see [6, Theorem 2.6]). We show in Example 4.3 that an extension of a compact group by a reflexive precompact (even ω-bounded) group may be non-reflexive. In Corollary 3.5 we prove, in contrast, that an extension of a reflexive P-group by another reflexive P-group is always reflexive.

Answering a question in [20] we prove in Section 5 that, while the P-modification of an LCA group is always reflexive, the P-modification of a reflexive σ-compact group can fail to be reflexive.

1.1. Notation and terminology

All groups considered here are assumed to be Abelian. A *character* of a topological group G is a continuous homomorphism of G to the circle group $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$ when the latter is considered as a subgroup of in the complex plane $\mathbb{C}$ with its usual topology and multiplication. The group $G^\wedge$ of all characters of G with the pointwise multiplication is called the *dual group* or simply the *dual* of G. The dual group $G^\wedge$ carries the *compact-open* topology τ_{co} defined as follows.

Put $\mathbb{T}_+ = \{z \in \mathbb{T}\colon \mathrm{Re}(z) \geqslant 0\}$. For a non-empty set $K \subset G$, we define

$$K^\triangleright = \{\chi \in G^\wedge\colon \chi(K) \subset \mathbb{T}_+\}.$$

The collection of sets $\{K^\triangleright\colon K \subset G,\ K \text{ is compact}\}$ forms a neighborhood basis at the identity of $G^\wedge$ for the compact-open topology τ_{co}. Let us note that the sets $K^\triangleright$ are not necessarily open in $(G^\wedge, \tau_{co})$ since $\mathbb{T}_+$ is not open in $\mathbb{T}$. If, instead of $\mathbb{T}_+$, we use a smaller neighborhood U of 1 in $\mathbb{T}$ to construct the sets $K^\triangleright$, the resulting sets will also form a neighborhood basis at the identity for the compact-open topology τ_{co}.

The subgroup

$$K^\perp = \{\chi \in G^\wedge\colon \chi(K) = \{1\}\}$$