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A note on local bases and convergence in fuzzy metric spaces

Valentín Gregori [*,1], Juan-José Miñana [2], Samuel Morillas [3]

Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia, Spain

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ABSTRACT

In the context of fuzzy metrics in the sense of George and Veeramani, we study when certain families of open balls centered at a point are local bases at this point. This question is related to p-convergence and s-convergence.

1. Introduction

George and Veeramani, [1,3], introduced and studied a notion of fuzzy metric with the help of continuous t-norms. If M is a fuzzy metric on a (non-empty) set X, a topology τ_M is deduced from M. In [2,8] the authors show that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable spaces. Later, several authors have contributed to the development of this theory, for instance [8,10,11,16–18].

An interesting aspect in this type of fuzzy metric is that it includes in its definition a parameter t. This feature has been successfully used in engineering applications such color image filtering [7,14,15] and perceptual color differences [5,13]. From the mathematical point of view it allows to introduce novel (fuzzy metric) concepts that only have natural sense in the fuzzy metric setting. For instance, the concept of p-convergence [4] and s-convergence [6] of sequences, which satisfy the implications

$$s\text{-}convergence \quad \Rightarrow \quad convergence \quad \Rightarrow \quad p\text{-}convergence.$$

* Corresponding author. Tel./fax: +034 962849300.
E-mail addresses: vgregori@mat.upv.es (V. Gregori), juamiapr@upvnet.upv.es (J.-J. Miñana), smorillas@mat.upv.es (S. Morillas).

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The convergence of a sequence to a point x_0 in a metric space (X,d) involves some local base constituted by balls centered at x_0. If ξ is any family of open balls centered at x_0 such that $\bigcap\xi = \{x_0\}$ and x_0 is not isolated in (X,d) then ξ is a local base at x_0. (In this paper $\bigcap\xi$ denotes the intersection of all members of ξ.) The purpose of this note is to study this assertion in the fuzzy setting. We consider first, a general case, and later some families of balls that, in a natural way, appear when studying p-convergence and s-convergence. Notice that a centered ball at x_0 in a fuzzy metric space $(X,M,*)$ is denoted by $B(x_0,r,t)$ where $r \in]0,1[$, $t > 0$.

We show in this paper that the above assertion is false, in general, for a fuzzy metric space (Example 7). Now, if ξ is constituted by balls of the form $\{B(x_0,r,r)\colon r \in J\}$, where $J \subset]0,1[$, or M is stationary (Definition 2) then the above assertion holds.

In [4] it is proved that any sequence p-convergent to x_0 in $(X,M,*)$ is convergent if and only if $\{B(x_0,r,t)\colon r \in]0,1[\}$ is a local base at x_0, for each $t > 0$. Fuzzy metric spaces in which all p-convergent sequences are convergent were called principal. So it seems natural to study families of open balls, centered at x_0, for a fixed $t > 0$. We show that if $\mathcal{B}$ is any of these families the above assertion is true in principal fuzzy metric spaces, but in general it is false.

In [6] it is proved that any sequence convergent to x_0 is s-convergent in $(X,M,*)$ if and only if $\bigcap_{t>0} B(x_0,r,t)$ is a neighborhood of x_0 in (X,τ_M), for each $r \in]0,1[$. Fuzzy metric spaces in which all convergent sequences are s-convergent were called s-fuzzy metric spaces. So, it is natural to study families of open balls centered at x_0 with a fixed radius $r \in]0,1[$. If $\mathcal{D}$ is any of these families the above assertion is true in co-principal fuzzy metric spaces (Definition 19), and a similar result is obtained when (X,τ_M) is compact (Theorem 23). The answer in a more general context is an open problem (Problem 17). Some examples are provided, along the paper, that illustrate the theory.

The structure of the paper is as follows. In Section 2 we include the preliminaries about fuzzy metrics. In Section 3 we study the question of when a family ξ of open balls centered at x_0 in a (principal) fuzzy metric space $(X,M,*)$, is a local base at x_0 provided that $\bigcap\xi = \{x_0\}$. The same question related to s-fuzzy metrics is studied in Section 4.

2. Preliminaries

Definition 1. (George and Veeramani [1]) A fuzzy metric space is an ordered triple $(X,M,*)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0,\infty[$ satisfying the following conditions, for all $x,y,z \in X$, $s,t > 0$:

(GV1) $M(x,y,t) > 0$;
(GV2) $M(x,y,t) = 1$ if and only if $x = y$;
(GV3) $M(x,y,t) = M(y,x,t)$;
(GV4) $M(x,y,t) * M(y,z,s) \leqslant M(x,z,t+s)$;
(GV5) $M(x,y,\_) :]0,\infty[\to]0,1]$ is continuous.

If $(X,M,*)$ is a fuzzy metric space, we will say that $(M,*)$, or simply M, is a *fuzzy metric* on X. This terminology will be also extended along the paper in other concepts, as usual, without explicit mention.

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x,\epsilon,t)\colon x \in X,\ 0 < \epsilon < 1,\ t > 0\}$, where $B_M(x,\epsilon,t) = \{y \in X\colon M(x,y,t) > 1-\epsilon\}$ for all $x \in X$, $\epsilon \in]0,1[$ and $t > 0$, and they proved that for each $x \in X$ the family $\{B(x,\frac{1}{n},\frac{1}{n})\colon n \in \mathbb{N}\}$ is a local base at x. A sequence $\{x_n\}$ in (X,τ_M) converges to $x \in X$ if and only if $\lim_n M(x_n,x,t) = 1$ for all $t > 0$. Also, in [1] the authors defined the closed ball $B_M[x,r,t] = \{y \in X\colon M(x,y,t) \geqslant 1\}$ and proved that it is a closed set in τ_M. If confusion is not possible we write simply B instead of B_M.