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A domain-theoretic approach to fuzzy metric spaces

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ABSTRACT

We introduce a partial order $\sqsubseteq_M$ on the set $\mathbf{B}X$ of formal balls of a fuzzy metric space $(X, M, \wedge)$ in the sense of Kramosil and Michalek, and discuss some of its properties. We also characterize when the poset $(\mathbf{B}X, \sqsubseteq_M)$ is a continuous domain by means of a new notion of fuzzy metric completeness introduced here. The well-known theorem of Edalat and Heckmann that a metric space is complete if and only if its poset of formal balls is a continuous domain, is deduced from our characterization.

1. Introduction and preliminaries

Throughout this paper the letter $\mathbb{N}$ will denote the set of all positive integer numbers.

Let us recall [19] that a continuous t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $a * 1 = a$ for all $a \in [0,1]$; (iii) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$, for all $a, b, c, d \in [0,1]$; (iv) $*$ is continuous on $[0,1] \times [0,1]$.

Typical examples of continuous t-norms are the minimum, denoted by $\wedge$, and the product denoted by $\cdot$, i.e., $a \wedge b = \min\{a, b\}$ and $a \cdot b = ab$ for all $a, b \in [0,1]$. It is well known and easy to see that $* \leqslant \wedge$ for any continuous t-norm $*$.

Definition 1. ([11, Definition 7]) A fuzzy metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a function from $X \times X \times [0, \infty)$ to $[0,1]$, such that for all $x, y, z \in X$:

(KM1) $M(x, y, 0) = 0$;
(KM2) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;

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(KM3) $M(x,y,t) = M(y,x,t)$;
(KM4) $M(x,z,t+s) \geqslant M(x,y,t) * M(y,z,s)$ for all $t,s \geqslant 0$;
(KM5) $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous.

A fuzzy metric space is a triple $(X,M,*)$ such that X is a set and $(M,*)$ is a fuzzy metric on X.

From (KM2) and (KM4) it follows that for all $x,y \in X$, $M(x,y,.)$ is a non-decreasing function.

Each fuzzy metric $(M,*)$ on a set X induces a topology τ_M on X which has as a base the family of open balls $\{B_M(x,\varepsilon,t)\colon\ x \in X,\ \varepsilon \in (0,1),\ t > 0\}$, where $B_M(x,\varepsilon,t) = \{y \in X\colon\ M(x,y,t) > 1-\varepsilon\}$.

If $(x_n)_n$ is a sequence in $(X,M,*)$ which converges to a point $x \in X$ with respect to τ_M, we shall write $\lim_{n\to\infty} x_n = x$. Observe that $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} M(x,x_n,t) = 1$ for all $t > 0$.

A sequence $(x_n)_n$ in a fuzzy metric space $(X,M,*)$ is called a Cauchy sequence if for each $t > 0$ and each $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\varepsilon$ for all $n,m \geqslant n_0$. $(X,M,*)$ is said to be complete if every Cauchy sequence converges with respect to τ_M (see e.g. [5]).

Remark 1. It is well known (see e.g. [8]) that every (complete) fuzzy metric space $(X,M,*)$ is (completely) metrizable, i.e., there exists a (complete) metric d on X whose induced topology coincides with τ_M. Conversely, if (X,d) is a (complete) metric space and we define $M_d : X \times X \times [0,\infty) \to [0,1]$ by $M_d(x,y,0) = 0$ and

$$M_d(x,y,t) = \frac{t}{t + d(x,y)},$$

for all $t > 0$, then $(X,M_d,\wedge)$ is a (complete) fuzzy metric space called the standard fuzzy metric space of (X,d) (compare [4,5]). Moreover, the topology τ_{M_d} coincides with the topology induced by d.

Next we recall several concepts from the theory of domains which will be useful later on (see e.g. [6]).

A partial order on a set X is a reflexive, antisymmetric and transitive relation $\sqsubseteq$ on X. In this case, we say that the pair $(X,\sqsubseteq)$ is a partially ordered set (a poset, in short).

An element x of a poset $(X,\sqsubseteq)$ is called maximal if condition $x \sqsubseteq y$ implies $x = y$. The set of maximal elements of $(X,\sqsubseteq)$ will be denoted by $\mathrm{Max}((X,\sqsubseteq))$.

A subset D of a poset $(X,\sqsubseteq)$ is directed provided that it is non-empty and any pair of elements of D has an upper bound in D. The least upper bound of a subset D of X is denoted by $\bigsqcup D$ if it exists. A poset $(X,\sqsubseteq)$ is directed complete, and is called a dcpo, if every directed subset of $(X,\sqsubseteq)$ has a least upper bound.

Let x and y be two elements of a poset $(X,\sqsubseteq)$. We say that x is way below y, in symbols $x \ll y$, if for each directed subset D of $(X,\sqsubseteq)$ for which $\bigsqcup D$ exists, the relation $y \sqsubseteq \bigsqcup D$ implies $x \sqsubseteq z$ for some $z \in D$. A poset $(X,\sqsubseteq)$ is continuous if for each $x \in X$, the set $\Downarrow\!x = \{u \in X\colon\ u \ll x\}$ is directed, and $x = \bigsqcup(\Downarrow\!x)$. A continuous poset which is also a dcpo is called a continuous domain, or simply, a domain if no confusion arises.

In this paper we are interested in the problem of establishing relationships between the theory of complete fuzzy metric spaces and domain theory. Our study is motivated, in part, by the previous researches about the construction of computational models for metric spaces and other related structures by using domains (see the New notes of Chapter V-6 in [6] and the references given therein. See also [1–3,10,12,14–18,20,21]). In particular, Lawson [12] proved that a metric space is a maximal point space if and only if it is complete and separable. Later on, Edalat and Heckmann [2] established, in a nice and explicit way, several connections between (complete) metric spaces and domain theory by using the notion of a (closed) formal ball.

A formal ball in a (non-empty) set X is simply a pair (x,r), with $x \in X$ and $r \in [0,\infty)$. The set of formal balls of X is the Cartesian product $X \times [0,\infty)$ which will be denoted by $\mathbf{B}X$ in the sequel.