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# Subgroups of products of paratopological groups

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ABSTRACT

We give a characterization of the paratopological groups that can be topologically embedded as subgroups into a product of first-countable (second-countable) $T_i$ paratopological groups for $i = 0, 1$. We show that a $T_1$ paratopological group $G$ admits a topological embedding as a subgroup into a topological product of first-countable (second-countable) $T_1$ paratopological groups if and only if $G$ is $\omega$-balanced (totally $\omega$-narrow) and the *symmetry number* of $G$ is countable, i.e., for every neighborhood $U$ of the identity $e$ in $G$ we can find a countable family $\gamma$ of neighborhoods of $e$ satisfying $\bigcap_{V\in\gamma} V^{-1} \subseteq U$. We show that every 2-pseudocompact $T_1$ paratopological group with a countable symmetry number is a topological group. We answer in the negative some questions posed by Manuel Sanchis and Mikhail Tkachenko by constructing an example of a commutative functionally Hausdorff totally $\omega$-narrow paratopological group of countable pseudocharacter $H$ such that there is no continuous isomorphism from $H$ onto a Hausdorff first-countable paratopological group. The group $H$ is not topologically isomorphic to a subgroup of a product of Hausdorff second-countable paratopological groups.



## 1. Introduction

A *paratopological* (*semitopological*) group is a group endowed with a topology for which multiplication in the group is jointly (separately) continuous. If, additionally, the inversion in a paratopological group is continuous, then it is called a *topological group*.

There are many results on the question of when a paratopological (semitopological) group is in fact a topological group. For example, in 1957, R. Ellis showed that each locally compact Hausdorff semitopological group is a topological group [5]. Later, Żelazko showed that every completely metrizable semitopological group is a topological group [14]. In 1996, Bouziad proved in [4] that every Čech-complete semitopological group is a topological group.

In 2007, M. Sanchis and M. Tkachenko showed that every Hausdorff paratopological group $G$ is a topological group provided that $G$ is a Lindelöf P-space [10]. We show in Proposition 2.6 that "Hausdorff" in this result can be weakened to "$T_1$". Thus, if $G$ is a $T_1$ paratopological group such that $G$ is a Lindelöf P-space, then $G$ is a topological group.

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In 2010, Ravsky proved that every 2-pseudocompact paratopological group of countable pseudocharacter is a topological group [8]. This result answers a question posed by Alas and Sanchis (see [1, Question B]). In Proposition 2.9 we give necessary and sufficient conditions under which a 2-pseudocompact $T_1$ paratopological group turns out to be a topological group. This characterization implies Ravsky's result mentioned above.

According to [12], given a topological property $\mathcal{P}$, we say that a paratopological (topological) group $G$ is *projectively* $\mathcal{P}$ if for every neighborhood $U$ of the identity in $G$, there exists a continuous homomorphism $p\colon G \to H$ onto a paratopological (topological) group $H$ with property $\mathcal{P}$ such that $p^{-1}(V) \subseteq U$, for some neighborhood $V$ of the neutral element in $H$. It is easy to see that a $T_i$ paratopological group is projectively first-countable (second-countable) $T_i$ if and only if it admits a homeomorphic embedding as a subgroup into a product of first-countable (second-countable) $T_i$ paratopological groups, for $i = 0, 1, 2, 3, 3\frac{1}{2}$.

Katz showed that a topological group is topologically isomorphic to a subgroup of a topological product of first-countable topological groups if and only if it is $\omega$-balanced [7]. The case of second-countable topological groups is completely described by I. Guran in [6]: A topological group is topologically isomorphic to a subgroup of the product of some family of second-countable topological groups if and only if it is $\omega$-narrow.

Tkachenko gives an internal characterization of the projectively Hausdorff first-countable (second-countable) paratopological groups using the cardinal invariant called the *Hausdorff number* (see [12, Theorems 2.7–2.8]). The case of regular paratopological groups is described in [12, Theorems 3.6, 3.8]: A regular paratopological group $H$ can be topologically embedded as a subgroup into a product of regular first-countable (second-countable) paratopological groups if and only if $H$ is $\omega$-balanced (totally $\omega$-narrow) and has countable *index of regularity*.

Motivated by the techniques used in [12], we characterize subgroups of topological products of families of first-countable (second-countable) $T_i$ paratopological groups for $i = 0, 1$. We prove in Theorems 2.17 and 2.19 that a $T_0$ paratopological group is projectively first-countable (second-countable) $T_0$ if and only if it is $\omega$-balanced (totally $\omega$-narrow). We also show in Theorems 2.20 and 2.22 that a $T_1$ paratopological group $G$ can be topologically embedded as a subgroup into a product of $T_1$ first-countable (second-countable) paratopological groups if and only if $G$ is $\omega$-balanced (totally $\omega$-narrow) and has a countable *symmetry number*.

M. Sanchis and M. Tkachenko posed the following question (see [10, Problem 6.3]): Suppose that $H$ is a Hausdorff totally $\omega$-narrow paratopological group of countable pseudocharacter. Does $H$ admit a continuous isomorphism onto a Hausdorff second-countable paratopological group? We answer this question in Example 2.28 in the negative by constructing a commutative functionally Hausdorff totally $\omega$-narrow paratopological group of countable pseudocharacter $H$ such that there is no continuous isomorphism from $H$ onto a Hausdorff first-countable paratopological group. The group $H$ has uncountable Hausdorff number, so Example 2.28 provides an answer (again in the negative) to the question posed by Tkachenko in [12, Problem 4.1]: Does every totally $\omega$-narrow Hausdorff paratopological group $G$ have countable Hausdorff number?

## 2. Symmetry number

Denote by $\mathcal{N}(e)$ the family of neighborhoods of the identity $e$ in a semitopological group $G$. A semitopological group $G$ is $T_1$ if and only if for each $x \in G \setminus \{e\}$, there exists $V \in \mathcal{N}(e)$ such that $x \notin V$ or, equivalently, $\bigcap_{V\in\mathcal{N}(e)} V^{-1} = \{e\}$. Motivated by this observation, we define the *symmetry number* of a $T_1$ semitopological group $G$, denoted by $Sm(G)$, as the minimum cardinal number $\kappa$ such that for every neighborhood $U$ of $e$ in $G$, there exists a family $\gamma \subseteq \mathcal{N}(e)$ such that $\bigcap_{V\in\gamma} V^{-1} \subseteq U$ and $|\gamma| \leqslant \kappa$.

A semitopological group $G$ satisfies $Sm(G) = 1$ if and only if $G$ is a *quasitopological group*, i.e., a semitopological group with continuous inversion. The following three propositions are immediate consequences from the definition of the symmetry number (we suppose that all spaces are $T_1$).