Topology and its Applications 169 (2014) 2–15

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Joint metrizability of subspaces and perfect mappings

A.V. Arhangel'skii [a,b], M.M. Choban [c,*], M.A. Al Shumrani [b]

[a] Moscow 121165, Russia
[b] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[c] Department of Mathematics, Tiraspol State University, Kishinev, Republic of Moldova

ARTICLE INFO

MSC:
primary 54A25
secondary 54B05

Keywords:
k-Leader
k-Space
Compactly metrizable
Countably metrizable
JPM-space
Perfect mapping
Sequential
Quotient mapping
Submetrizable

ABSTRACT

Following a general idea in [6,7], we introduce and study in this paper the concept of a *JPM*-space. We call in this way topological spaces admitting a metric which metrizes every metrizable subspace of X. It is shown that any Hausdorff sequential *JPM*-space is metrizable. It is also proved that perfect mappings with metrizable fibers preserve the class of Hausdorff *JPM*-spaces. Some new results concerning countably metrizable spaces and compactly metrizable spaces introduced in [6] and [7] are also obtained.

1. Introduction

Let X be a topological space, and $\mathcal{F}$ be a family of subspaces of X. Following [6,7], we will say that X is *jointly metrizable on* $\mathcal{F}$, or $\mathcal{F}$*-metrizable*, if there is a metric d on the set X such that d *metrizes all subspaces of* X *which belong to* $\mathcal{F}$, that is, the restriction of d to A generates the subspace topology on A, for any $A \in \mathcal{F}$. In particular, X is *compactly metrizable*, or X *is jointly metrizable on compacta*, if X is jointly metrizable on all compact subspaces of X [6]. We also consider the case when $\mathcal{F}$ is the family of all metrizable subspaces of X. Then $\mathcal{F}$-metrizable spaces are called *JPM*-spaces. Notice that every extremally disconnected space is a *JPM*-space. A space X is called *countably metrizable* if it is jointly metrizable on the family of all countable subspaces.

It is shown that any Hausdorff sequential *JPM*-space is metrizable. It is also proved that perfect mappings with metrizable fibers preserve the class of Hausdorff *JPM*-spaces. Curiously, the condition that fibers are metrizable cannot be dropped, since every compact Hausdorff space can be represented as an image of an

* Corresponding author.
E-mail addresses: arhangel.alex@gmail.com (A.V. Arhangel'skii), mmchoban@gmail.com (M.M. Choban), maalshmrani1@kau.edu.sa (M.A. Al Shumrani).

http://dx.doi.org/10.1016/j.topol.2014.02.027
0166-8641/

extremally disconnected compactum under a perfect mapping. This theorem is one of our main results. It follows from it that if a Hausdorff space X is covered by a locally finite family of closed JPM-subspaces, then X is a JPM-space as well. We also establish certain connections between various types of $\mathcal{F}$-metrizability and some classical topological properties, one of which is metrizability. In terminology and notation we follow [8] and [5].

The *tightness* $t(X)$ of a topological space X is *countable*, if for an arbitrary subset A of X and any x in the closure of A, there is a countable subset B of A such that $x \in \overline{B}$. A space X is *sequential*, if for any non-closed subset A of X there exists a point $x \in X \setminus A$ such that some sequence of points of A converges to x. A space X is called *m-sequential*, if a subset P of X is closed in X whenever the intersection of P with any metrizable compact subspace F of X is closed in the subspace F. Below a space X is called a *k-space*, if a subset P of X is closed in X whenever the intersection of P with any compact subspace F of X is closed in the subspace F. Usually, k-spaces are assumed to be Hausdorff, but we drop this assumption from the definition.

Every topological space X is jointly metrizable on the family of all discrete subspaces of X. Also if $\gamma = \{X_\alpha\colon\ \alpha \in A\}$ is a disjoint family of metrizable subspaces of a space X, then X is jointly metrizable on γ.

Below is a sample of results on spaces jointly metrizable on compacta and on spaces jointly metrizable on countable subspaces obtained in [6] and [7].

Example 1.1. ([7]) Take an uncountable set T with the cocountable topology $\mathcal{T}_c$ in which the only closed sets are countable sets and the set T. The space T is not Hausdorff. On the other hand, every countable subspace of X is closed and discrete. Therefore, all compact subspaces and all metrizable subspaces of T are discrete. Hence, T is jointly metrizable on the family of all compact subspaces, all countable subspaces and all metrizable subspaces.

Theorem 1.2. *([6]) Every submetrizable space X is jointly metrizable on compacta.*

Since the Sorgenfrey line is submetrizable, it is jointly metrizable on compacta, while it is not metrizable. Notice that the Sorgenfrey line is first-countable and hence, is a k-space. Every Tychonoff space with a countable network is jointly metrizable on compacta, while it needn't be metrizable. In particular, for any uncountable separable Tychonoff space X, the space $C_p(X)$ is a natural example of a non-metrizable space which is jointly metrizable on compacta. If f is a closed continuous mapping of a metrizable space X onto a space Y, then Y is jointly metrizable on compacta.

Theorem 1.3. *([6,7]) A k-space X is jointly metrizable on compacta if and only if it is submetrizable.*

For each topological space $(X, \mathcal{T})$, we define a new topology $\mathcal{T}_k$ on X by the rule: a subset U of X belongs to $\mathcal{T}_k$ if and only if the set $U \cap F$ is open in F, for every compact subspace F of $(X, \mathcal{T})$. The topological space $(X, \mathcal{T}_k)$ is called the *k-leader* of the space $(X, \mathcal{T})$ and is denoted by kX (see [1]).

Corollary 1.4. *([7]) A space X is jointly metrizable on compacta if and only if its k-leader kX is submetrizable.*

Not every submetrizable Tychonoff space is countably metrizable.

Example 1.5. ([6]) Take any non-metrizable countable Tychonoff space X. Any such space is submetrizable. By Theorem 1.2, X is jointly metrizable on compacta. On the other hand, X is not countably metrizable. Thus, metrizability on compacta does not imply countable metrizability.