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On ∞-convex sets in spaces of scatteredly continuous functions

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Dedicated to Mitrofan Choban and Stoyan Nedev on the occasion of their 70th birthdays

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ABSTRACT

Given a topological space X, we study the structure of ∞-convex subsets in the space $SC_p(X)$ of scatteredly continuous functions on X. Our main result says that for a topological space X with countable strong fan tightness, each potentially bounded ∞-convex subset $\mathcal{F} \subset SC_p(X)$ is weakly discontinuous in the sense that each non-empty subset $A \subset X$ contains an open dense subset $U \subset A$ such that each function $f|U$, $f \in \mathcal{F}$, is continuous. This implies that $\mathcal{F}$ has network weight $nw(\mathcal{F}) \leqslant nw(X)$.

1. Introduction

In this paper we continue the study of the linear-topological structure of the function spaces $SC_p(X)$, started in [7,8,4]. Here $SC_p(X)$ stands for the space of all scatteredly continuous real-valued functions on a topological space X, endowed with the topology of pointwise convergence. A function $f : X \to Y$ between two topological spaces is called *scatteredly continuous* if for each non-empty subset $A \subset X$ the restriction $f|A$ has a point of continuity.[2] By [2] or [3, 4.4], each scatteredly continuous function $f : X \to Y$ with values in a regular topological space Y is *weakly discontinuous* in the sense that each non-empty subset $A \subset X$ contains an open dense subset $U \subset A$ such that the restriction $f|U$ is continuous. This implies that $SC_p(X)$ is a linear space.

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The space $SC_p(X)$ contains the linear subspace $C_p(X)$ of all continuous real-valued functions on X. By its topological properties the function spaces $SC_p(X)$ and $C_p(X)$ differ substantially. For example, for an uncountable metrizable separable space X, the function space $SC_p(X)$ contains non-metrizable compacta while $C_p(X)$ does not. In spite of this difference, in [4] it was observed that compact *convex* subsets of the space $SC_p(X)$ share many properties with compact convex subsets of the space $C_p(X)$. In particular, for a space X with countable network weight, all compact convex subsets of the function space $SC_p(X)$ are metrizable. This follows from Corollary 1 of [4] saying that for a countably tight topological space X, each σ-convex subset $\mathcal{F} \subset SC_p(X)$ has network weight $nw(\mathcal{F}) \leqslant nw(X)$.

A subset K of a linear topological space L is called *σ-convex* if for any sequence $(x_n)_{n\in\omega}$ of points of K and any sequence $(t_n)_{n\in\omega}$ of non-negative real numbers with $\sum_{n=0}^{\infty} t_n = 1$ the series $\sum_{t=0}^{\infty} t_n x_n$ converges to a point of the set K. It is easy to see that each compact convex subset K of a locally convex linear topological space L is σ-convex, and each σ-convex subset of L is convex and bounded in L. We recall that a subset B of a linear topological space L is *bounded* in L if for each neighborhood $U \subset L$ of zero there is a number $n \in \mathbb{N}$ such that $B \subset nU$.

A more general notion than the σ-convexity is the ∞-convexity. Following [5], we define a subset K of a linear topological space L to be *∞-convex* if for each bounded sequence $(x_n)_{n\in\omega}$ in K and each sequence $(t_n)_{n\in\omega}$ of non-negative real numbers with $\sum_{n=0}^{\infty} t_n = 1$ the series $\sum_{t=0}^{\infty} t_n x_n$ converges to a point of the set K. In Proposition 5.1 we shall prove that a subset K of a linear topological space L is σ-convex if and only if it is bounded and ∞-convex.

In this paper we shall study the structure of ∞-convex subsets in the spaces $SC_p(X)$ of scatteredly continuous functions on a topological space X. A principal question, which will be asked about ∞-convex subsets $\mathcal{F} \subset SC_p(X)$ is their weak discontinuity. A function family $\mathcal{F} \subset SC_p(X)$ is called *weakly discontinuous* if each subset $A \subset X$ contains a dense open subset $U \subset A$ such that for each function $f \in \mathcal{F}$ the restriction $f|U$ is continuous. It is easy to see that a function family $\mathcal{F} \subset SC_p(X)$ is weakly discontinuous if and only if its diagonal product $\Delta\mathcal{F} : X \to \mathbb{R}^{\mathcal{F}}$, $\Delta\mathcal{F} : x \mapsto (f(x))_{f\in\mathcal{F}}$, is weakly discontinuous.

In [4] we proved that for each countably tight space X, each σ-convex subset $\mathcal{F}$ of the space $SC_p^*(X)$ of bounded scatteredly continuous functions on X is weakly discontinuous and has network weight $nw(\mathcal{F}) \leqslant nw(X)$. Our first theorem is the generalization of this fact to ∞-convex subsets of the function space $SC_p^*(X)$.

Theorem 1.1. *If a topological space X has countable tightness, then each ∞-convex subset $\mathcal{F} \subset SC_p^*(X)$ is weakly discontinuous and has network weight $nw(\mathcal{F}) \leqslant nw(X)$.*

Let us recall that a topological space X has *countable tightness* if for each subset $A \subset X$ and a point $a \in \bar{A}$ in its closure there is an at most countable subset $B \subset A$ with $a \in \bar{B}$.

Observe that Theorem 1.1 treats ∞-convex subsets of the function space $SC_p^*(X)$. For topological spaces X with countable strong fan tightness, this theorem remains true for potentially bounded ∞-convex subsets of the function space $SC_p(X)$.

Following [12] we shall say that a topological space X has *countable strong fan tightness* if for each sequence $(A_n)_{n\in\omega}$ of subsets of X and a point $a \in \bigcap_{n\in\omega} \bar{A}_n$ there is a sequence of points $a_n \in A_n$, $n \in \omega$, such that a lies in the closure of the set $\{a_n\}_{n\in\omega}$. The class of spaces of countable strong fan tightness and some related classes will be discussed in Section 8.

A subset $B \subset L$ of a linear topological space L is called *potentially bounded* if for each sequence $(x_n)_{n\in\omega}$ of points of B there is a sequence $(t_n)_{n\in\omega}$ of positive real numbers such that the set $\{t_n x_n\}_{n\in\omega}$ is bounded in L. For example, the set $SC_p^*(X)$ is potentially bounded in the function space $SC_p(X)$.

Theorem 1.2. *If a topological space X has countable strong fan tightness, then each potentially bounded ∞-convex subset $\mathcal{F} \subset SC_p(X)$ is weakly discontinuous and has network weight $nw(\mathcal{F}) \leqslant nw(X)$.*