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Monotone countable paracompactness and maps to ordered topological vector spaces

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In this paper, we characterize monotonically countably paracompact (or monotonically countably metacompact) spaces by semi-continuous maps with values into some ordered topological vector spaces or some topological vector lattices. These extend earlier results for real-valued semi-continuous functions.

1. Introduction and preliminaries

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces. Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}$ the set of all natural numbers, κ an infinite cardinal, and ω the first infinite cardinal. (Topological) vector spaces always mean real (topological) vector spaces. C. Good, R. Knight and I. Stares [9] and C. Pan [22] introduced a monotone version of countably paracompact spaces, called monotonically countably paracompact spaces and monotonically cp-spaces, respectively, and it is proved in [9, Proposition 14] that both these notions are equivalent.

For two sequences $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$ of subsets of a space, it is written that $(A_n) \preccurlyeq (B_n)$ if $A_n \subset B_n$ for each $n \in \omega$.

Definition 1.1. ([9,22]) A topological space X is said to be *monotonically countably metacompact* if there exists an operator U assigning to each decreasing sequence $(D_j)_{j\in\omega}$ of closed subsets of X with $\bigcap_{j\in\omega} D_j = \emptyset$, a sequence $U((D_j)) = (U(n,(D_j)))_{n\in\omega}$ of open subsets of X such that

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(1) $D_n \subset U(n, (D_j))$ for each $n \in \omega$;
(2) $\bigcap_{n\in\omega} U(n, (D_j)) = \emptyset$;
(3) If $(D_j) \preccurlyeq (E_j)$, then $U((D_j)) \preccurlyeq U((E_j))$.

X is said to be *monotonically countably paracompact* if, in addition,

(2′) $\bigcap_{n\in\omega} \overline{U(n, (D_j))} = \emptyset$.

This definition is due to C. Good, R. Knight and I. Stares [9]. See also [22] for an equivalent definition due to C. Pan, and [29] for various equivalent conditions of monotonically countably paracompact spaces and monotonically countably metacompact spaces due to C. Ying and C. Good. Note that monotonically countably metacompact spaces coincide with β-spaces [9, Theorem 5].

The following two theorems were proved in [8, Theorem 3], [9, Theorem 25] and [26, Theorem 2.4 and Corollary 3.3].

Theorem 1.2. *([8,9,26]) For a topological space X, the following conditions are equivalent*:

(1) *X is monotonically countably paracompact.*
(2) *There exists an operator Φ assigning to each locally bounded function $f : X \to \mathbb{R}$, a locally bounded lower semi-continuous function $\Phi(f) : X \to \mathbb{R}$ with $|f| \leqslant \Phi(f)$ such that $\Phi(f) \leqslant \Phi(f')$ whenever $|f| \leqslant |f'|$.*
(3) *There exist operators Φ, Ψ assigning to each upper semi-continuous function $f : X \to \mathbb{R}$ with $f \geqslant 0$, a lower semi-continuous function $\Phi(f) : X \to \mathbb{R}$ and an upper semi-continuous function $\Psi(f) : X \to \mathbb{R}$ with $f \leqslant \Phi(f) \leqslant \Psi(f)$ such that $\Phi(f) \leqslant \Phi(f')$ and $\Psi(f) \leqslant \Psi(f')$ whenever $f \leqslant f'$.*

Theorem 1.3. *([26]) For a topological space X, the following conditions are equivalent*:

(1) *X is monotonically countably metacompact.*
(2) *There exists an operator Φ assigning to each locally bounded function $f : X \to \mathbb{R}$, a lower semi-continuous function $\Phi(f) : X \to \mathbb{R}$ with $|f| \leqslant \Phi(f)$ such that $\Phi(f) \leqslant \Phi(f')$ whenever $|f| \leqslant |f'|$.*
(3) *There exists an operator Φ assigning to each upper semi-continuous function $f : X \to \mathbb{R}$ with $f \geqslant 0$, a lower semi-continuous function $\Phi(f) : X \to \mathbb{R}$ with $f \leqslant \Phi(f)$ such that $\Phi(f) \leqslant \Phi(f')$ whenever $f \leqslant f'$.*

The purpose of this paper is to generalize real-valued functions in Theorems 1.2 and 1.3 to maps with values into some ordered topological vector spaces Y. This provides some advantage to the real-valued cases. Indeed, the range $\mathbb{R}$ with the total order can be extended to spaces Y with the partial order. Also, our results do not make use of basic tools depending on Banach space theories or selection principles.

In the rest of this section, let us recall some definitions and terminology from [24], see also [1,2,7]. For a topological space X and $A \subset X$, $\operatorname{int}_X A$ and $\overline{A}$ denote the interior and the closure, respectively, of A in X.

The origin of a vector space is denoted by $\mathbf{0}$. Let Y be a vector space. For $y \in Y$ and $A, B \subset Y$, define $-A = \{-a\colon a \in A\}$, $A + B = \{a + b\colon a \in A, b \in B\}$, $A - B = A + (-B)$, and $y + A = \{y\} + A$. Note that the complement of A in Y $(= \{y \in Y\colon y \notin A\})$ is denoted by $Y \setminus A$. A subset $A \subset Y$ is *circled* if $\lambda A \subset A$ whenever $|\lambda| \leqslant 1$. For a map $f : X \to Y$, the map $-f : X \to Y$ is defined by $(-f)(x) = -f(x)$ for each $x \in X$. The map $0 : X \to Y$ is defined by $0(x) = \mathbf{0}$ for each $x \in X$.

A partially ordered vector space $(Y, \leqslant)$ is said to be an *ordered vector space* if the following conditions are satisfied: