Topology and its Applications 169 (2014) 99–107

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Asymptotic dimension, decomposition complexity, and Haver's property C

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ARTICLE INFO

MSC:
primary 54F45
secondary 53C23

Keywords:
Property C
Asymptotic property C
Finite decomposition complexity (FDC)
Straight FDC

ABSTRACT

The notion of the decomposition complexity was introduced in [14] using a game theoretic approach. We introduce a notion of straight decomposition complexity and compare it with the original as well with the asymptotic property C. Then we define a game theoretic analog of Haver's property C in the classical dimension theory and compare it with the original.

1. Introduction

Asymptotic dimension was introduced by Gromov to study finitely generated groups though the definition can be applied for all metric spaces [12]. It received a great deal of attention when Goulyang Yu proved the Novikov higher signature conjecture for groups with finite asymptotic dimension [23]. There were many other similar results about groups and manifolds under assumption of finiteness of the asymptotic dimension or the asymptotic dimension of the fundamental group [6,2,4,8]. When G. Yu introduced property A and proved the coarse Baum–Connes for groups with property A, it was a natural problem to check whether every finitely presented group has this property. A construction of a finitely presented group without property A was suggested by Gromov [13] (see for detailed presentation [1]). Gromov's random group construction is an existence theorem. Still, it is a good question whether a given group (or class of groups) has property A. The answer is unknown for the Thompson group F.

It turns out that the dimension theoretic approach to verification of property A proved to be quite productive. There are different extensions of features of finite asymptotic dimension to asymptotically infinite dimensional spaces. Some of them came from the analogy with classical dimension theory, some from function growth, and some from game theory. The asymptotic property C was defined in [5] by

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1 Supported by NSF grant DMS-0904278.

http://dx.doi.org/10.1016/j.topol.2014.02.035
0166-8641/

analogy with Haver's property C in the classical dimension theory (see also [19]). It was shown that the asymptotic property C implies the property A. Different versions of asymptotic dimension growth were suggested in [7,3,9,10]. The property A was proven for groups with the sublinear dimension growth [5,3], then for the polynomial dimension growth [7], and finally, for the subexponential dimension growth [18]. Since dimension growth of a finitely generated group is at most exponential, this leaves a question whether the property A is equivalent to a subexponential dimension growth.

The notion of decomposition complexity of a metric space was introduced in [14] (see also [17]) in game theoretic terms. It was shown that the finite decomposition complexity (FDC) implies property A. The finite decomposition complexity was verified for a large class of groups, in particular for all countable subgroups of $GL(n, K)$ for an arbitrary field K [14]. In this paper we address the question what is the relation between asymptotic property C and FDC. It turns out that in the classical dimension theory there is no analog of FDC. We plan to present one and compare it with Haver's property C in a future publication. Here to make a comparison of FDC and the asymptotic property C we introduce the notion of the straight finite decomposition complexity sFDC opposed to the game theoretic finite decomposition complexity = gFDC = FDC. We prove the following implications

$$gFDC \quad \Rightarrow \quad sFDC \quad \Rightarrow \quad property\ A$$

and

$$asymptotic\ property\ C \quad \Rightarrow \quad sFDC.$$

We know that the last implication is not an equivalence for metric spaces. We expect that it is not an equivalence for groups as well (see Question 4.3). We do not know if any of the first two implications is reversible.

In Section 5 of the paper we compare a game-theoretic approach with the standard in the classical dimension theory. We did the comparison for Haver's property C and found that a game-theoretic analog of it defines the countable dimensionality. It is known that these classes are different [11].

2. Preliminaries

All spaces are assumed to be metrizable.

A generic metric is denoted by d. Given two nonempty subsets A, B of X, we let $d(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$.

Let $R > 0$. We say that a family $\mathcal{A}$ of nonempty subsets of X is R-disjoint if $d(A, B) > R$, for every $A, B \in \mathcal{A}$.

A metric space X is geodesic if for every $x, y \in X$ there exists an isometric embedding $\alpha\colon [0, d(x, y)] \to X$ such that $\alpha(0) = x$ and $\alpha(d(x, y)) = y$.

A metric space X is discrete if there exists $C > 0$ such that $d(x, y) \geqslant C$, for every $x, y \in X$, $x \neq y$. A discrete metric space is said to be of bounded geometry if there exists a function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that every ball of radius r contains at most $f(r)$ points.

Let $\mathcal{X}$, $\mathcal{Y}$ be families of metric spaces and $R > 0$. We say that $\mathcal{X}$ is R-decomposable over $\mathcal{Y}$ if, for any $X \in \mathcal{X}$, $X = \bigcup(\mathcal{V}_1 \cup \mathcal{V}_2)$, where $\mathcal{V}_1$, $\mathcal{V}_2$ are R-disjoint families and $\mathcal{V}_1 \cup \mathcal{V}_2 \subset \mathcal{Y}$.

A family $\mathcal{X}$ of metric spaces is said to be *bounded* if

$$\operatorname{mesh}(\mathcal{X}) = \sup\{\operatorname{diam} X \mid X \in \mathcal{X}\} < \infty.$$

Let $\mathfrak{A}$ be a collection of metric families. A metric family $\mathcal{X}$ is *decomposable over* $\mathfrak{A}$ if, for every $r > 0$, there exists a metric family $\mathcal{Y} \in \mathfrak{A}$ and an r-decomposition of $\mathcal{X}$ over $\mathcal{Y}$.