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Hereditarily normal Wijsman hyperspaces are metrizable ☆

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ABSTRACT

In this paper, we study normality and metrizability of Wijsman hyperspaces. We show that every hereditarily normal Wijsman hyperspace is metrizable. This provides a partial answer to a problem of Di Maio and Meccariello in 1998.

1. Introduction

Throughout this paper, let 2^X denote the family of all nonempty closed subsets of a given topological space X. For a metric space (X,d), let $d(x,A) = \inf\{d(x,a)\colon a \in A\}$ denote the distance between a point $x \in X$ and a nonempty subset A of (X,d), and $S_d(A,\varepsilon) = \{x \in X\colon d(x,A) < \varepsilon\}$ is called the *ε-enlargement* of A. A net $\{A_\alpha\colon \alpha \in D\}$ in 2^X is said to be *Wijsman convergent to* some A in 2^X if $d(x,A_\alpha) \to d(x,A)$ for every $x \in X$. The Wijsman topology on 2^X induced by d, denoted by $\tau_{w(d)}$, is the weakest topology such that for every $x \in X$, the distance functional $d(x,\cdot)\colon 2^X \to \mathbb{R}^+$ is continuous. To see the structure of this topology, for any $E \subseteq X$, let $E^- = \{A \in 2^X\colon A \cap E \neq \emptyset\}$. It can be seen easily that the Wijsman topology

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on 2^X induced by d has the family

$$\{U^-: U \text{ is open in } X\} \cup \{\{A \in 2^X: d(x,A) > \varepsilon\}: x \in X, \varepsilon > 0\}$$

as a subbase, [2]. Moreover, for a finite subset $E \subseteq X$, $A \in 2^X$ and $\varepsilon > 0$, let

$$\mathcal{N}_{A,E,\varepsilon} = \{B \in 2^X: |d(x,A) - d(x,B)| < \varepsilon \text{ for all } x \in E\}.$$

Then for any $A \in 2^X$, the collection

$$\{\mathcal{N}_{A,E,\varepsilon}: E \subseteq X \text{ is finite and } \varepsilon > 0\}$$

forms a neighborhood base of A in $\tau_{w(d)}$.

The *Wijsman hyperspace* of a metric space (X,d) is the space $(2^X, \tau_{w(d)})$, and in turn, (X,d) is called the *base space* of $(2^X, \tau_{w(d)})$. Sometimes, the shorter notation $2^{(X,d)}$ is used for the hyperspace $(2^X, \tau_{w(d)})$. If $\mathcal{H} \subseteq 2^X$, for the notational convenience, we still use the symbol $\tau_{w(d)}$, rather than $\tau_{w(d)}|_{\mathcal{H}}$, to denote the relative Wijsman topology on $\mathcal{H}$.

The above type of convergence was introduced by Wijsman in [21] for sequences of closed convex sets in Euclidean space $\mathbb{R}^n$, when he considered optimum properties of the sequential probability ratio test. In [17], Wijsman convergence was used in the general framework of a metric space, and the metrizability of the Wijsman topology of a separable metric space was established. Since then, there has been a considerable effort to explore various topological properties of Wijsman hyperspaces. For example, Beer [1] and Costantini [8] studied Polishness of Wijsman hyperspaces, Cao and Tomita [6] as well as Zsilinszky [22] investigated Baireness of Wijsman hyperspaces, Cao and Junnila [4] studied Amsterdam properties of Wijsman spaces. However, Wijsman hyperspaces are far to be completely understood, and still there are many problems concerning fundamental properties of these objects unsolved. This motivates the authors to continue their study of Wijsman hyperspaces in the present paper.

Note that all Wijsman topologies are Tychonoff, since they are weak topologies. In a more recent paper, Cao, Junnila and Moors [5] showed that Wijsman hyperspaces are universal Tychonoff spaces in the sense that every Tychonoff space is embeddable as a closed subspace in the Wijsman hyperspace of a complete metric space which is locally $\mathbb{R}$. Thus, one of the fundamental problems is to determine when a Wijsman hyperspace is normal. The problem was first mentioned by Di Maio and Meccariello in [9], where it was asked whether the normality of a Wijsman hyperspace is equivalent to its metrizability. A partial solution to this problem, which asserts that the answer is "yes" when the base space of a Wijsman hyperspace is a normed linear space, was recently observed by Holá and Novotný in [13]. The main purpose of this paper is to give another partial answer to this problem. By using techniques similar to those of Keesling in [15], we are able to establish that a Wijsman hyperspace is hereditarily normal if and only if it is metrizable.

The rest of this paper is organized as follows. In Section 2, an overview on the normality and metrizability of basic types of hyperspaces is provided. The main result and its proof are given in Section 3. Our terminology and notation are standard. For undefined terms, refer to [2], [3] or [10].

2. Normality and metrizability of hyperspaces

It has been an interesting and challenging problem in general topology to characterize normality of the hyperspace of a topological space. In 1955, Ivanova [14] showed that $2^{\mathbb{N}}$ with the Vietoris topology is not normal, where $\mathbb{N}$ is equipped with the discrete topology. Continuing in this direction, Keesling [15] proved that under the CH (Continuum Hypothesis), for a Tychonoff space X, $(2^X, \tau_V)$ is normal if and only if $(2^X, \tau_V)$ is compact (and thus if and only if X is compact), where τ_V denotes the Vietoris topology on 2^X. In addition, he also showed in [16] that for a regular T_1 space X, a number of covering properties of