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Generalized homogeneity and weakly Klebanov spaces ☆

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ABSTRACT

It is shown that three results of A.V. Arhangel'skii for homogeneous spaces (see [2]) hold for *co*-homogeneous spaces. For instance, if a *co*-homogeneous space X contains a G_δ-dense weakly Klebanov subspace, then X is weakly Klebanov.

1. Introduction

The present paper is a continuation of the investigations from the articles [2,4,5] and [6]. By a space we understand a Tychonoff topological space. We use the terminology from [7].

Recall that a space X is called *homogeneous* if for any two points $a, b \in X$ there exists a homeomorphism $h_{ab} : X \to X$ such that $h_{ab}(a) = b$.

More precisely the following generalizations, defined in [4], of the notion of homogeneity are considered.

A space X is called:

- *lo-homogeneous* if for any two points $a, b \in X$ there exist two open subsets U and V of X and a continuous open mapping $h_{ab} : U \to V$ such that $a \in U$, $b \in V$ and $h_{ab}(a) = b$;
- *co-homogeneous* if for any two points $a, b \in X$ there exist two open subsets U and V of X and a continuous open mapping $h_{ab} : U \to V$ such that $a \in U$, $b \in V$, $h_{ab}(a) = b$ and the set $cl_X h_{ab}^{-1}(x)$ is countably compact for each $x \in V$;
- *do-homogeneous* if for any two points $a, b \in X$ there exist two open subsets U and V of X, two subsets A and B and a continuous open mapping $h_{ab} : A \to B$ such that $a \in A \subseteq U \subseteq cl_X A$, $b \in B = h_{ab}(A) \subseteq V \subseteq cl_X B$ and $h_{ab}(a) = b$.

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The following examples show that this generalizations of homogeneity are distinct from one another:

Example 1.1. (Arhangel'skii, Choban and Mihaylova [4]) Let C be the unit circle, $\mathbb{R}$ be the space of reals and X be the discrete sum of the spaces C and $\mathbb{R}$. The space X is not homogeneous and is *co*-homogeneous.

Example 1.2. (Arhangel'skii, Choban and Mihaylova [4]) Let C be the unit circle, J be the space of irrationals, $Y = C \times J$ and $X = Y \oplus J$ be the discrete sum of the spaces Y and J. Obviously, the spaces C, J and Y are homogeneous. The space X is not homogeneous. It will be affirmed that the space X is *co*-homogeneous. Fix two points $a, b \in X$. Only the following case will be considered $a \in J$ and $b = (b_1, b_2) \in Y$.

Fix a homeomorphism $f : J \to J$ such that $f(b_2) = a$. The mapping $\varphi : Y \to J$, where $\varphi(x, y) = f(y)$ for every point $(x, y) \in Y = C \times J$ is open and perfect. By construction $\varphi(b) = a$. Every separable complete metrizable space without isolated points is an open continuous image with compact fibers of the space of irrationals J (see [1]). Hence, there exists a continuous open mapping with compact fibers $\psi : J \to Y$ such that $\psi(a) = b$. Now consider the open continuous mapping with compact fibers $h_{ab} : X \to X$ such that $h_{ab}|J = \psi$, $h_{ab}|Y = \varphi$, $h_{ab}(a) = b$ and $h_{ab}(b) = a$.

Example 1.3. (Arhangel'skii, Choban and Mihaylova [4]) Let $\mathbb{R}$ be the space of reals, J be the space of irrationals, $Y = \mathbb{R} \times J$ and $X = Y \oplus J$ be the discrete sum of the spaces Y and J. Obviously, the spaces $\mathbb{R}$, J and Y are homogeneous. The space X is not homogeneous. The space X is *lo*-homogeneous. Fix $b \in J$ and $a = (a_1, a_2) \in Y$. Let U and V be open subsets of X, $b \in V \subseteq J$ and $a = (a_1, a_2) \in U \subseteq Y$, $h : U \to V$ be an open continuous mapping of U onto V and $h(a) = b$. One can assume that $U = (a_1 - r, a_1 + r) \times W$, where $r > 0$ and W is an open subset of J. The set $F = U \cap (\mathbb{R} \times \{a_2\}) = (a_1 - r, a_1 + r) \times \{a_2\}$ is not compact and it is closed in the subspace U. Since the set F is connected and the space J is zero-dimensional, it follows $h(F) = b$. Since F is a closed subset of the set $h^{-1}(b)$, the set $h^{-1}(b)$ is not compact. Moreover, the set $h^{-1}(y)$ is not compact for any $y \in V$. Hence, the space X is not *co*-homogeneous.

Example 1.4. Let C be the unit circle, J be the space of irrationals and $X = C \oplus J$ be the discrete sum of the spaces C and J. If $a, b \in X$, then there exists a countable dense subspace Z of the space X for which $a, b \in Z$. By virtue of Sierpinski's theorem ([10], or [7, Exercise 6.2A.d]), the space Z is homeomorphic to the space of rationals. Therefore the space Z is homogeneous and there exists a homeomorphism $h_{ab} : Z \to Z$ such that $h_{ab}(a) = (b)$. Hence X is a *do*-homogeneous space. The space X is not locally compact and points of the subspace C are points of local compactness in X. Since a *lo*-homogeneous space with points of local compactness is locally compact, the space X is not *lo*-homogeneous.

Recall that a space X is called *d-homogeneous* (see [3]) if for every two points $a, b \in X$ there exist two dense subspaces A and B of X and a homeomorphism $h_{ab} : A \to B$ such that $a \in A$, $b \in B$ and $h_{ab}(a) = b$. The space X from Example 1.4 is d-homogeneous.

2. Weakly Klebanov spaces and Malychin spaces

A space X is called weakly Klebanov if the closure of the union of any family of G_δ-subsets of X is also the union of a family of G_δ-subsets of X (see [2]).

Weakly Klebanov spaces were defined by A.V. Arhangel'skii in order to investigate power homogeneous spaces and he has shown how this property works in a much more efficient way than the Moscow property. Clearly, every Klebanov space is a weakly Klebanov space (see [8]). In fact, every space of countable pseudocharacter is weakly Klebanov though it need not be a Klebanov space.