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Selections, paraconvexity and PF-normality ☆

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ABSTRACT

We prove a selection theorem for paraconvex-valued mappings defined on τ-PF normal spaces. The method developed to prove this result is used to provide a general approach to such selection theorems.

1. Introduction

Let X and Y be topological spaces, and let 2^Y be the family of all non-empty subsets of Y. Also, let

$$\mathscr{F}(Y) = \{S \in 2^Y\colon\ S \text{ is closed}\} \quad \text{and} \quad \mathscr{C}(Y) = \{S \in \mathscr{F}(Y)\colon\ S \text{ is compact}\}.$$

A set-valued mapping $\varphi : X \to 2^Y$ is *lower semi-continuous*, or *l.s.c.*, if the set

$$\varphi^{-1}(U) = \{x \in X\colon\ \varphi(x) \cap U \neq \varnothing\}$$

is open in X for every open $U \subset Y$. A single-valued mapping $f : X \to Y$ is a *selection* for $\varphi : X \to 2^Y$ if $f(x) \in \varphi(x)$ for every $x \in X$.

Let Y be a normed space. Throughout this paper, we will use d to denote the metric on Y generated by the norm of Y. Following [9], a subset P of Y is called *α-paraconvex*, where $0 \leqslant \alpha \leqslant 1$, if whenever $r > 0$ and $d(p, P) < r$ for some $p \in Y$, then

$$d(q, P) \leqslant \alpha r \quad \text{for all } q \in \operatorname{conv}(B_r(p) \cap P).$$

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Here, $B_r(x) = \{y \in Y\colon\ d(x,y) < r\}$, and $\mathrm{conv}(A)$ is the *convex hull* of A. The set P is called *paraconvex* if it is α-paraconvex for some $\alpha < 1$. A closed set is 0-paraconvex if and only if it is convex.

In the sequel, τ will denote an infinite cardinal number, and $w(Y)$ — the topological weight of the space Y. Also, we will use $\mathscr{C}'(Y) = \mathscr{C}(Y) \cup \{Y\}$.

In [8], E. Michael proved that if X is paracompact and Y is a Banach space, then every l.s.c. convex-valued mapping $\varphi : X \to \mathscr{F}(Y)$ has a continuous selection (see [8, Theorem 3.2″]). In [9], E. Michael generalized this result by replacing "convexity" with "α-paraconvexity" for a fixed $\alpha < 1$ (see [9, Theorem 2.1]); this generalization remains valid for τ-paracompact normal spaces, see [6, Theorem 3.2]. P.V. Semenov generalized Michael's paraconvex-valued selection theorem by replacing the constant α with a continuous function $f : (0,\infty) \to [0,1)$ satisfying a certain property called (PS) (*functional paraconvexity*, see [22]); and D. Repovš and P.V. Semenov considered in [14] a function $\alpha_P : (0,\infty) \to [0,2)$ (called the *function of nonconvexity*) associated to each nonempty subset $P \subset Y$, see [14,22] for the definition of these concepts. Also, they obtained several applications of selections for paraconvex-valued mappings, see [15–19,23] and the monograph [20]. The author has recently proved a τ-collectionwise normal version of these results, i.e. when X is τ-collectionwise normal, Y is a Banach space with $w(Y) \leqslant \tau$, and φ is α-paraconvex- and $\mathscr{C}'(Y)$-valued [6, Theorem 2.1]. Let us explicitly remark that the proofs of these theorems utilize the fact that τ-paracompactness and τ-collectionwise normality are hereditary with respect to closed subsets.

We are now ready to state also the main purpose of this paper. Namely, we prove a paraconvex-valued selection theorem for $\mathscr{C}(Y)$-valued mappings defined on τ-PF-normal spaces (Corollary 3.3), see Section 3 for the definitions of these spaces. The challenge in this particular case is that τ-PF-normality is not hereditary with respect to closed subsets, hence the method used for the τ-collectionwise normal spaces in [6] cannot be applied straightforward; the rest of the arguments are similar. In fact, using the property of mappings discussed in the next section, we prove a slightly more general result (Theorem 3.1) and derive from a common point of view all previous known results of paraconvex-valued selection theorems for l.s.c. mappings (see Examples 2.2 and 2.3).

2. The Dense Multi-selection Property

For $\varepsilon > 0$, a single-valued mapping $g : X \to Y$ to a metric space (Y,d) is an *ε-selection* for $\varphi : X \to 2^Y$, if $d(g(x),\varphi(x)) < \varepsilon$, for every $x \in X$. Also, a set-valued mapping $\psi : X \to 2^Y$ is called a *set-valued selection* (or a *multi-selection*) for another set-valued mapping $\varphi : X \to 2^Y$ if $\psi(x) \subset \varphi(x)$, for every $x \in X$. We shall say that a mapping $\varphi : X \to 2^Y$ has the *Dense Multi-selection Property*, or *DMP* for short, where (Y,d) is a metric space, if the following hold:

(i) φ has an l.s.c. multi-selection $\psi : X \to \mathscr{C}(Y)$.

(ii) For every $\varepsilon > 0$, a cozero-set $U \subset X$, and a continuous ε-selection $g : U \to Y$ for $\varphi \restriction U$, there exists an l.s.c. $\psi : U \to \mathscr{C}(Y)$ such that

$$\psi(x) \subset \overline{\varphi(x) \cap B_\varepsilon\big(g(x)\big)}, \quad x \in U.$$

We may consider open balls $B_\varepsilon(y)$ when $\varepsilon = \infty$. Thus, $B_\infty(y) = Y$, and the DMP of $\varphi : X \to 2^Y$ can be simply expressed by saying that for every $0 < \varepsilon \leqslant \infty$, a cozero-set $U \subset X$, and a continuous ε-selection $g : U \to Y$ for $\varphi \restriction U$, there exists an l.s.c. $\psi : U \to \mathscr{C}(Y)$ such that $\psi(x) \subset \overline{\varphi(x) \cap B_\varepsilon(g(x))}$, $x \in U$.

Remark 2.1. In the realm of normal spaces, cozero-sets coincide with open F_σ-sets. So, if X is normal, $U \subset X$ is an open F_σ-set, and $\varphi : X \to \mathscr{F}(Y)$ has the DMP, then $\varphi \restriction U$ will automatically have the DMP.