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On representation spaces

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A B S T R A C T

Let $\mathcal{C}$ be a class of topological spaces, let $\mathcal{P}$ be a subset of $\mathcal{C}$, and let α be a class of mappings having the composition property. Given $X \in \mathcal{C}$, we write $X \in \mathrm{Cl}_\alpha(\mathcal{P})$ if for every open cover $\mathcal{U}$ of X there is a space $Y \in \mathcal{P}$ and a $\mathcal{U}$-mapping $f : X \to Y$ that belongs to α. The closure operator Cl_α defines a topology τ_α in $\mathcal{C}$. After proving general properties of the operator Cl_α, we investigate some properties of the topological space $(\mathbb{N}, \tau_\alpha)$, where $\mathbb{N}$ is the space of all nondegenerate metric continua and α is one of the following classes: all mappings, confluent mappings, or monotone mappings.

1. Introduction

In continuum theory many results can be expressed using ε-mappings. The subject have been started by P.S. Aleksandrov in [1], then continued by K. Kuratowski, S. Ulam, S. Eilenberg, and other in the thirties of the twentieth century (see for example [16] and [11]). Let us recall an important result by S. Mardešić and J. Segal (see [18]) that every continuum admits ε-mappings onto polyhedra.

Note that if $\mathcal{P}$ is a class of ANRs the existence of ε-mappings onto members of $\mathcal{P}$ is equivalent to the fact that a continuum X can be expressed as an inverse limit of a sequence of elements of $\mathcal{P}$ with surjective bonding mappings. Therefore one can define arc-like, circle-like, tree-like, disk-like, etc., using ε-mappings

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or, equivalently, using inverse limits. This is not true anymore if the class $\mathcal{P}$ does not consist of ANRs, or if we restrict the class of mappings. In fact, there is not much known about when the existence of ε-mappings is enough for the existence of an appropriate inverse limit if we restrict to some classes of mappings. The only article on the subject is [21].

Interesting results and problems can be obtained if we restrict the class of mappings to a class of confluent, monotone, or another class with composition property (see [3] or [21]). Recently, a lot of results about ε-mappings and ε-properties have been gathered in [3].

In general topology one needs to use $\mathcal{U}$-mappings, for a given cover $\mathcal{U}$ of the domain, instead of ε-mappings.

In this article we observe that one can introduce a natural closure operator and investigate the space of some topological spaces with the topology generated by that operator. This is another point of view on continuum theory or on topology in general. Some old results can be expressed in the language of the topological space $(\mathcal{C}, \tau_\alpha)$, but also some new theorems and problems arise.

2. Definitions, notation and basic results

Definition 2.1. Given two topological spaces X and Y and a cover $\mathcal{U}$ of X, we say that a mapping $f : X \to Y$ is a $\mathcal{U}$-mapping if there is an open cover $\mathcal{V}$ of Y such that $\{f^{-1}(V)\colon\ V \in \mathcal{V}\}$ refines $\mathcal{U}$.

Definition 2.2. Let $\mathcal{C}$ be a class of topological spaces and let α be a class of mappings between elements of $\mathcal{C}$. We say that α has the composition property if

(1) for every $X \in \mathcal{C}$ the identity map $id_X : X \to X$ is in α,
(2) if $f : X \to Y$ and $g : Y \to Z$ are in α, then $g \circ f$ is in α.

Definition 2.3. Let $\mathcal{C}$ be a class of topological spaces, let $\mathcal{P}$ be a subset of $\mathcal{C}$, and let α be a class of mappings having the composition property. Given $X \in \mathcal{C}$, we write $X \in \mathrm{Cl}_\alpha(\mathcal{P})$ if for every open cover $\mathcal{U}$ of X there is a space $Y \in \mathcal{P}$, and a $\mathcal{U}$-mapping $f : X \to Y$ that belongs to α.

Theorem 2.4. *The operator* Cl_α *satisfies the following Kuratowski axioms of the closure operator* (*see* [15, §4, p. 38]):

(1) $\mathcal{A} \subset \mathrm{Cl}_\alpha(\mathcal{A})$,
(2) $\mathrm{Cl}_\alpha(\mathcal{A}) = \mathrm{Cl}_\alpha(\mathrm{Cl}_\alpha(\mathcal{A}))$,
(3) $\mathrm{Cl}_\alpha(\mathcal{A} \cup \mathcal{B}) = \mathrm{Cl}_\alpha(\mathcal{A}) \cup \mathrm{Cl}_\alpha(\mathcal{B})$,
(4) $\mathrm{Cl}_\alpha(\emptyset) = \emptyset$.

Proof. In order to show (1), it is enough to put $Y = X$, to take f as the identity mapping on X, and put $\mathcal{V} = \mathcal{U}$.

To show (2), first observe that (1) and (2) imply that $\mathrm{Cl}_\alpha(\mathcal{A}) \subseteq \mathrm{Cl}_\alpha(\mathrm{Cl}_\alpha(\mathcal{A}))$. To see the other inclusion, take $X \in \mathrm{Cl}_\alpha(\mathrm{Cl}_\alpha(\mathcal{A}))$ and an open cover $\mathcal{U}$ of X. Then, by the definition, there is a space $Y \in \mathrm{Cl}_\alpha(\mathcal{A})$, a mapping $f : X \to Y$ in α, and a cover $\mathcal{V}$ of Y such that $\{f^{-1}(V)\colon\ V \in \mathcal{V}\}$ refines $\mathcal{U}$. Again, since $Y \in \mathrm{Cl}_\alpha(\mathcal{A})$, there is a space $Z \in \mathcal{A}$, a mapping $g : Y \to Z$, $g \in \alpha$, and an open cover $\mathcal{W}$ of Z such that $\{g^{-1}(V)\colon\ V \in \mathcal{W}\}$ refines $\mathcal{V}$. To see that $X \in \mathrm{Cl}_\alpha(\mathcal{A})$, it is enough to consider the space Z, the mapping $g \circ f$, that is in α by the composition property of α, and the cover $\mathcal{W}$.

To show (3), first observe that (1) and (2) imply that $\mathrm{Cl}_\alpha(\mathcal{A}) \cup \mathrm{Cl}_\alpha(\mathcal{B}) \subseteq \mathrm{Cl}_\alpha(\mathcal{A} \cup \mathcal{B})$. To see the other inclusion, take $X \in \mathrm{Cl}_\alpha(\mathcal{A} \cup \mathcal{B}) \setminus \mathrm{Cl}_\alpha(\mathcal{A})$. We need to show that $X \in \mathrm{Cl}_\alpha(\mathcal{B})$. Since $X \notin \mathrm{Cl}_\alpha(\mathcal{A})$, there is an open cover $\mathcal{U}_0$ of X such that for every space $Y \in \mathcal{A}$, for every $f : X \to Y$ satisfying $f \in \alpha$, and for every cover $\mathcal{V}$ of Y, the family $\{f^{-1}(V)\colon\ V \in \mathcal{V}\}$ does not refine $\mathcal{U}_0$. To show that $X \in \mathrm{Cl}_\alpha(\mathcal{B})$, consider an open