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Behavior of the Eilenberg–Moore spectral sequence in derived string topology

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A B S T R A C T

The purpose of this paper is to give applications of the Eilenberg–Moore type spectral sequence converging to the relative loop homology algebra of a Gorenstein space, which is introduced in the previous paper due to the authors. Moreover, it is proved that the spectral sequence is functorial on the category of simply-connected Poincaré duality spaces over a space.

1. Introduction

This is a sequel to the paper [12]. In the previous paper, we have developed a general theory of derived string topology, namely string topology on Gorenstein spaces due to Félix and Thomas [7]. One of machineries in derived string topology is the Eilenberg–Moore spectral sequence (EMSS) converging to the loop homology of a Gorenstein space. This paper aims at making explicit computations of relative loop homology algebras of Poincaré duality spaces by employing the EMSS. Moreover, we establish the functoriality of the EMSS on appropriate categories.

In what follows, the coefficients of the (co)homology and the singular cochain algebra of a space are in a field $\mathbb{K}$ unless otherwise explicitly stated. Moreover, it is assumed that spaces have the homotopy type of CW-complexes whose homologies with coefficients in an underlying field are of finite type.

Let $f : N \to M$ be a map. By definition, the relative loop space $L_f M$, for which we may write $L_N M$, fits into the pull-back diagram

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$$\begin{array}{ccc} L_f M & \longrightarrow & M^I \\ \big\downarrow & & \big\downarrow{\scriptstyle (ev_0, ev_1)} \\ N & \xrightarrow[(f,f)]{} & M \times M, \end{array}$$

where ev_t stands for the evaluation map at t. Suppose that N is a simply-connected Poincaré duality space. Then the so-called loop product on $H_*(L_N M)$ makes the shifted homology $\mathbb{H}_*(L_N M) := H_{*+\dim N}(L_N M)$ into an associative and unital algebra; see [12, Remark 2.6 and Proposition 2.7] and Proposition 3.5 below. We denote by $\mathbb{H}_*(LM)$ the relative loop homology $\mathbb{H}_*(L_M M)$ if $f : M \to M$ is the identity map. Observe that $\mathbb{H}_*(LM)$ is nothing but the loop homology due to Chas and Sullivan [1] when M is a closed oriented manifold; see [7]. We see that the product on $\mathbb{H}_*(LM)$ is an extension of the *intersection product* on the shifted homology $\mathbb{H}_*(M) := H_{*+d}(M)$ even if M is a Poincaré duality space; see Proposition 3.1 and the argument at the beginning of Section 3.

The following theorem is a particular version of [12, Theorem 2.11].

Theorem 1.1. *Let N be a simply-connected Poincaré duality space of dimension d. Let $f : N \to M$ be a continuous map to a simply-connected space M. Then the Eilenberg–Moore spectral sequence is a right half-plane cohomological spectral sequence $\{\mathbb{E}_r^{*,*}, d_r\}$ converging to the Chas–Sullivan loop homology $\mathbb{H}_*(L_N M)$ as an algebra with*

$$\mathbb{E}_2^{*,*} \cong HH^{*,*}\big(H^*(M); H^*(N)\big)$$

as a bigraded algebra; that is, there exists a decreasing filtration $\{F^p\mathbb{H}_*(L_N M)\}_{p \geqslant 0}$ of $\mathbb{H}_*(L_N M)$ such that $\mathbb{E}_\infty^{*,*} \cong Gr^{*,*}\mathbb{H}_*(L_N M)$ as a bigraded algebra, where

$$Gr^{p,q}\mathbb{H}_*(L_N M) = F^p\mathbb{H}_{-(p+q)}(L_N M)/F^{p+1}\mathbb{H}_{-(p+q)}(L_N M).$$

Here $HH^{*,*}(H^*(M), H^*(N))$ denotes the Hochschild cohomology with the cup product.

The original version of the theorem above is applicable to Gorenstein spaces whose class contains the classifying spaces of connected Lie groups, Noetherian H-spaces, homotopy quotients of closed oriented manifolds by compact Lie groups, Poincaré duality spaces and hence closed oriented manifolds; see [5,21,14]. In this paper, we introduce an explicit calculation of the relative loop homology of a Poincaré duality space over a space.

In general, it is difficult to compute the Chas–Sullivan loop homology $\mathbb{H}_*(LM)$ because the shifted homology is not functorial with respect to a map between Poincaré duality spaces. On the other hand, an important feature of the relative version of the loop homology is that it gives rise to a functor between appropriate categories. This is explained below.

Let $\mathbf{Poincaré}_M$ be the category of simply-connected based Poincaré duality spaces over M and based maps; that is, a morphism from $\alpha_1 : N_1 \to M$ to $\alpha_2 : N_2 \to M$ is a based map $f : N_1 \to N_2$ with $\alpha_1 = \alpha_2 \circ f$. Let $\mathbf{Top}_1^N$ be the category of simply-connected spaces under N. We denote by $\mathbf{GradedAlg}_A$ and $\mathbf{GradedAlg}^A$ the categories of unital graded algebras over an algebra A and of those under A, respectively. Assume that N is a simply-connected Poincaré duality space. Then, as mentioned above, the loop homology $\mathbb{H}_*(L_f M) := H_{*+\dim s(f)}(L_f M)$ comes with the loop product, where $s(f) = N$. In consequence, our consideration in [12] permits us to deduce the following theorem.

Theorem 1.2. (1) *The loop homology gives rise to functors*

$$\mathbb{H}_*(L_? M) := H_{*+\dim s(?)}(L_? M) : \mathbf{Poincaré}_M^{op} \to \mathbf{GradedAlg}_{H_*(\Omega M)}$$