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Topological size of scrambled sets for local dendrite maps ☆

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ABSTRACT

We prove that for any local dendrite map $f : X \to X$, any scrambled set of f is totally disconnected and hence it has empty interior. Moreover, we prove that scrambled sets are nowhere residual if the set of branch points of X is discrete, this holds in particular for Gehman dendrite.

1. Introduction

The study of topological size of scrambled sets is one of the most important tools to evaluate the complexity of the dynamics of a given map. First results in this subject were obtained by Bruckner and Hu for interval maps; they proved in [2] that scrambled sets are never residual. Mai in [5] showed that graph maps have only scrambled sets with empty interior. Blanchard et al. [1] then proved that in fact graph maps have only nowhere residual scrambled sets. However, there are examples of dendroid maps having scrambled sets with non-empty interior (see [8,3]). In this paper, we follow this line for the case of local dendrite maps. This covers both the graph and dendrite maps. Before stating our main results, we need some definitions and properties of local dendrites and local dendrite maps.

A continuum is a compact connected metric space. An arc is any space homeomorphic to the compact interval $[0,1]$. A topological space is arcwise connected if any two of its points can be joined by an arc.

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We use the terminology from Nadler [7]. By a *graph* we mean a continuum which can be written as the union of finitely many arcs such that any two of them are either disjoint or intersect only in one or both of their endpoints. A *tree* is a graph containing no subset homeomorphic to the circle.

By a *dendrite* X, we mean a locally connected continuum which contains no homeomorphic copy of a circle. Every sub-continuum of a dendrite is a dendrite ([7], Theorem 10.10) and every connected subset of X is arcwise connected ([7], Proposition 10.9). In addition, any two distinct points x, y of a dendrite X can be joined by a unique arc with end points x and y. We denote this arc by $[x,y]$ and put $[x,y) = [x,y] \setminus \{y\}$ (resp. $(x,y] = [x,y] \setminus \{x\}$ and $(x,y) = [x,y] \setminus \{x,y\}$).

By a *local dendrite* we mean a continuum such that each of its points has a closed neighborhood which is a dendrite. Every graph and every dendrite is a local dendrite. A local dendrite contains only finitely many circles ([4], Theorem 4, p. 303). Notice that the union of two graphs in a local dendrite with non-empty intersection is a graph. Let X be a local dendrite. For any arc A in X, we denote by $\gamma(A)$ the set of its endpoints. If $Y \subset X$ is a dendrite, then for any two distinct points $x, y \in Y$, we denote by $[x,y]_Y$ the unique arc in Y with end points x and y and put $[x,y)_Y = [x,y]_Y \setminus \{y\}$, $(x,y]_Y = [x,y]_Y \setminus \{x\}$ and $(x,y)_Y = [x,y]_Y \setminus \{x,y\}$). A point $x \in X$ is called a *branch point* of X if there exists a dendrite U which is a closed neighborhood of x, such that x is a branch point of U (i.e. $U \setminus \{x\}$ has more than two connected components). We denote by $B(X)$ the set of branch points of X.

Let $\mathbb{Z}_+$ and $\mathbb{N}$ be the sets of non-negative integers and positive integers respectively. Let X be a compact metric space with metric d and $f : X \longrightarrow X$ be a continuous map.

A pair (a,b) in $X \times X$ is called *proximal* if $\liminf_{n\to\infty} d(f^n(a), f^n(b)) = 0$. It is called *distal* if $\liminf_{n\to\infty} d(f^n(a), f^n(b)) > 0$ and it is called *asymptotic* if $\limsup_{n\to\infty} d(f^n(a), f^n(b)) = 0$. A pair (a,b) in $X \times X$ is called a *Li–Yorke* pair (of f) if it is proximal but not asymptotic. A subset S of X containing at least two points is called a *scrambled* set (of f) if any pair $(a,b) \in S^2$ with $a \neq b$, is a Li–Yorke pair. A continuous map from a local dendrite into itself is called a *local dendrite map*.

Our main results are the following:

Theorem 1.1. *If S is a scrambled set of a local dendrite map $f : X \to X$, then it is totally disconnected. In particular, S has empty interior.*

Theorem 1.2. *If $f : X \to X$ is a local dendrite map and $B(X)$ is discrete, then any scrambled set S of f is nowhere residual in X.*

2. Preliminaries

Lemma 2.1. *([1, p. 301]) Let X be a compact metric space and $f : X \to X$ be a continuous map. If $S \subset X$ is a scrambled set of f then, for any $n, k \in \mathbb{N}$:*

(i) *$f^n(S)$ is a scrambled set of f^k.*
(ii) *The restriction map $f^n_{|f^k(S)}$ is one-to-one.*

Lemma 2.2. *Let $f : X \to X$ be a local dendrite map. If J and $f(J)$ are two arcs in X such that $f(J) \subset J$ or $f(J) \supset J$, then J contains a fixed point of f.*

Lemma 2.3. *([6]) Let (X,d) be a local dendrite, then for every $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ such that, for any $x, y \in X$ with $d(x,y) \leqslant \delta$, there is an arc J with end points x and y such that* $\operatorname{diam}(J) < \varepsilon$.

Denote $\operatorname{diam}(A) := \sup_{x,y\in A} d(x,y)$ the diameter of a subset A of X.