Topology and its Applications 164 (2014) 113–124

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Satellites of an oriented surface link and their local moves

Inasa Nakamura *

Institute for Biology and Mathematics of Dynamical Cell Processes (iBMath), Interdisciplinary Center for Mathematical Sciences, Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan

A R T I C L E   I N F O

Article history:
Received 28 May 2013
Received in revised form 13 December 2013
Accepted 14 December 2013

MSC:
57Q45
57Q35

Keywords:
Surface link
2-dimensional braid
Roseman move
Chart

A B S T R A C T

For an oriented surface link F in $\mathbb{R}^4$, we consider a satellite construction of a surface link, called a 2-dimensional braid over F, which is in the form of a covering over F. We introduce the notion of an m-chart on a surface diagram $\pi(F) \subset \mathbb{R}^3$ of F, which is a finite graph on $\pi(F)$ satisfying certain conditions and is an extended notion of an m-chart on a 2-disk presenting a surface braid. A 2-dimensional braid over F is presented by an m-chart on $\pi(F)$. It is known that two surface links are equivalent if and only if their surface diagrams are related by a finite sequence of ambient isotopies of $\mathbb{R}^3$ and local moves called Roseman moves. We show that Roseman moves for surface diagrams with m-charts can be well-defined.

1. Introduction

A *surface link* is the image of an embedding of a closed surface into the Euclidean 4-space $\mathbb{R}^4$. Two surface links are *equivalent* if one is carried to the other by an ambient isotopy of $\mathbb{R}^4$. A *surface diagram* of a surface link is its projected generic image in $\mathbb{R}^3$ equipped with over/under information along each double point curve. It is known [19] that two surface links are equivalent if and only if their surface diagrams are related by a finite sequence of local moves called Roseman moves as illustrated in Fig. 1.1, and ambient isotopies of the diagrams in $\mathbb{R}^3$.

In this paper, we assume that surface links are oriented. We consider a satellite construction of a surface link F, which is in the form of a covering over F. In order to describe this construction, we introduce the notion of a 2-dimensional braid over F, which is an extended notion of a 2-dimensional braid over a 2-disk. A 2-dimensional braid over a 2-disk or a closed surface has a graphical presentation called an m-chart. We extend this notion to the notion of an m-chart on a surface diagram, and we show that a 2-dimensional braid over F is presented by an m-chart on its surface diagram (Theorem 5.5). Our main aim is to show

* Tel.: +81 3 5465 7000.
E-mail address: inasa@ms.u-tokyo.ac.jp.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.12.010

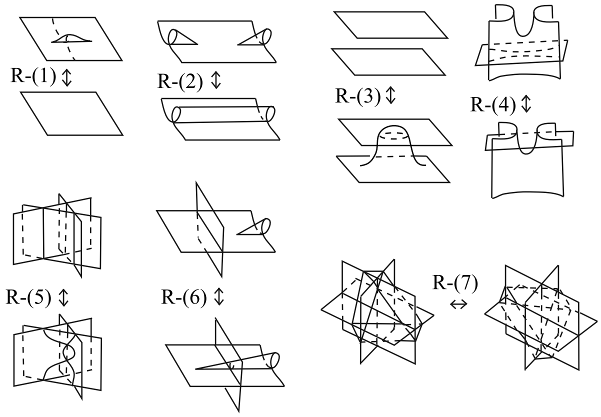

Fig. 1.1. Roseman moves. For simplicity, we omit the over/under information of each sheet.

that Roseman moves for surface diagrams with m-charts can be well-defined, by adding several new local moves to the original Roseman moves (Theorem 6.2).

The paper is organized as follows. In Section 2, we give the definition of 2-dimensional braids over a surface link. In Section 3, we review surface diagrams. In Section 4, we review chart description of 2-dimensional braids over a closed surface. In Section 5, we introduce the notion of an m-chart on a surface diagram and its presenting 2-dimensional braid, and we show Theorem 5.5. In Section 6, we introduce several new moves of Roseman moves for surface diagrams with m-charts. In Section 7, we give several remarks. Section 8 is devoted to showing Theorem 6.2.

2. Two-dimensional braids over a surface link

A 2-dimensional braid, which is also called a simple braided surface, over a 2-disk, is an analogous notion of a classical braid [7,11,20]. We can modify this notion to a 2-dimensional braid over a closed surface [17]. For a surface link F, we consider a closed surface embedded into $\mathbb{R}^4$ preserving the braiding structure as a satellite with companion F; see [1, Section 2.4.2]. We will call this a 2-dimensional braid over F.

Let Σ be a closed surface, let B^2 be a 2-disk, and let m be a positive integer.

Definition 2.1. A closed surface S embedded in $B^2 \times \Sigma$ is called a 2-*dimensional braid over* Σ of degree m if it satisfies the following conditions:

(1) The restriction $p|_S : S \to \Sigma$ is a branched covering map of degree m, where $p : B^2 \times \Sigma \to \Sigma$ is the projection to the second factor.
(2) For each $x \in \Sigma$, $\#(S \cap p^{-1}(x)) = m - 1$ or m.

Take a base point x_0 of Σ. Two 2-dimensional braids over Σ of degree m are *equivalent* if there is a fiber-preserving ambient isotopy of $B^2 \times \Sigma$ rel $p^{-1}(x_0)$ which carries one to the other.

A *trivial* 2-dimensional braid is the product of m distinct interior points of B^2 and Σ in $B^2 \times \Sigma$.

For a closed surface Σ, a surface link is said to be *of type* Σ when it is the image of an embedding of Σ. Let F be a surface link of type Σ, and let $N(F)$ be a tubular neighborhood of F in $\mathbb{R}^4$.

Definition 2.2. Let S be a 2-dimensional braid over Σ in $B^2 \times \Sigma$. Let $f : B^2 \times \Sigma \to \mathbb{R}^4$ be an embedding so that $f(B^2 \times \Sigma) = N(F)$. Then we call the image $f(S)$ a 2-*dimensional braid over* F. We call F, S, and f the *companion*, *pattern*, and *associated embedding* of the 2-dimensional braid, respectively. We define the *degree* of $f(S)$ as that of S.