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Bounds for fixed points on hyperbolic 3-manifolds

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ABSTRACT

For a compact hyperbolic 3-manifold M, we give bounds for the index for any homeomorphism $f: M \to M$ and any fixed point class $\mathbf{F}$ of f, $1 - 2\operatorname{rank}\pi_1(M) < \operatorname{ind}(f, \mathbf{F}) \leqslant 1$.

1. Introduction

Fixed point theory studies fixed points of a selfmap f of a space X. Nielsen fixed point theory, in particular, is concerned with the properties of the fixed point set

$$\operatorname{Fix} f := \{x \in X \mid f(x) = x\}$$

that are invariant under homotopy of the map f (see [3] for an introduction).

The fixed point set $\operatorname{Fix} f$ splits into a disjoint union of *fixed point classes*: two fixed points are in the same class if and only if they can be joined by a *Nielsen path* which is a path homotopic (relative to endpoints) to its own f-image. For each fixed point class $\mathbf{F}$, a homotopy invariant *index* $\operatorname{ind}(f, \mathbf{F}) \in \mathbb{Z}$ is defined. A fixed point class is *essential* if its index is non-zero.

A compact polyhedron X is said to have the *Bounded Index Property* (*BIP*) if there is an integer $B > 0$ such that for any map $f: X \to X$ and any fixed point class $\mathbf{F}$ of f, the index $|\operatorname{ind}(f, \mathbf{F})| \leqslant B$. X has the

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Bounded Index Property for Homeomorphisms (*BIPH*) if there is such a bound for all homeomorphisms $f: X \to X$.

The question we investigate is a problem of B. Jiang [4, Question 3]: Suppose a compact polyhedron X is aspherical (i.e. $\pi_i(X) = 0$ for all $i > 1$). Does X have BIP or BIPH?

In the paper [4], B. Jiang showed that graphs and surfaces with negative Euler characteristic have BIP (see [7] for an enhanced version). In the paper [9], McCord showed that infrasolv-manifolds have BIP. In the paper [5], B. Jiang and S. Wang showed that geometric 3-manifolds have BIPH for orientation preserving self-homeomorphisms: the index of each essential fixed point class is ± 1. In a recent paper [14], Q. Zhang showed that orientable Seifert 3-manifolds with hyperbolic orbifold have BIPH.

In this paper, we consider fixed point classes of homeomorphisms of hyperbolic 3-manifolds. We say that a compact 3-manifold M is *hyperbolic* if the interior $\operatorname{int} M$ of M admits a complete hyperbolic metric of finite volume. Then M is either closed or the boundary ∂M consisting of tori and Klein bottles (see [13, Theorem 5.11.1] or [1, Theorem 2.9]). All manifolds are assumed to be connected, unless it is specifically stated otherwise. For a group G, $\operatorname{rank} G$ denotes the minimal number of generators of G.

The main result of this paper is that hyperbolic 3-manifolds have BIPH:

Theorem 1.1. *Let M be a compact hyperbolic 3-manifold (orientable or nonorientable). Then M has BIPH. More precisely, if $f: M \to M$ is a homeomorphism, then*

(1) $\operatorname{ind}(f, \mathbf{F}) \leqslant 1$ *for every fixed point class* $\mathbf{F}$ *of* f;
(2) $\sum_{\operatorname{ind}(f,\mathbf{F})<0} \operatorname{ind}(f, \mathbf{F}) > 1 - 2\operatorname{rank}\pi_1(M)$, *where the sum is taken over all fixed point classes* $\mathbf{F}$ *with* $\operatorname{ind}(f, \mathbf{F}) < 0$.

2. Isometries of hyperbolic 3-manifolds

The aim of this section is to give some facts on isometries of hyperbolic 3-manifolds (see [6, §8]).

Suppose M is a compact hyperbolic 3-manifold, namely, the interior $\operatorname{int} M$ of M admits a complete hyperbolic metric of finite volume. Then we can identify M with the ϵ-thick part of a complete hyperbolic 3-manifold. More precisely, let M_ϵ be the submanifold of $\operatorname{int} M$ consisting of the points x such that there is an embedded hyperbolic open ball of radius ϵ centered at x. There exists an $\epsilon_0 > 0$ such that for any $\epsilon \leqslant \epsilon_0$, there is a homeomorphism $j: M \to M_\epsilon$, and each orientable boundary component of M_ϵ (if M is not closed) is a horosphere modulo a rank two abelian group of parabolic motions (see [13, 5.11]). We identify M with such an M_ϵ via j. Then M (not only $\operatorname{int} M$ when M has boundary) admits a hyperbolic metric pulled back from the hyperbolic metric of M_ϵ via j. Hence

Definition 2.1. Suppose M is a compact hyperbolic 3-manifold. A homeomorphism $f: M \to M$ is called an *isometry* if it preserves the hyperbolic metric of M pulled back from an ϵ-thick part M_ϵ of $\operatorname{int} M$ via j, namely,

$$j \circ f \circ j^{-1}: M_\epsilon \to M_\epsilon$$

is an isometry.

Lemma 2.2. *Suppose M is a compact hyperbolic 3-manifold and $f: M \to M$ is a homeomorphism. Then f can be homotopic to an isometry $g: M \to M$.*

Proof. Identify M with an ϵ-thick part M_ϵ of $\operatorname{int} M$ by a homeomorphism $j: M \to M_\epsilon$ and extend the homeomorphism $j \circ f \circ j^{-1}: M_\epsilon \to M_\epsilon$ to a homeomorphism $f': \operatorname{int} M \to \operatorname{int} M$. Since $\operatorname{int} M$ admits a complete hyperbolic metric of finite volume, by the well known Mostow Rigidity Theorem (see [11, p. 54]),