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Strong quasi-continuity of set-valued functions

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A B S T R A C T

By means of topological games, we will show that under certain circumstances on topological spaces X, Y and Z, every two variable set-valued function $F : X \times Y \to 2^Z$ is strongly upper (resp. lower) quasi-continuous provided that F_x is upper (resp. lower) semi-continuous and F^y is lower (resp. upper) quasi-continuous. Moreover, we will prove that if F is compact-valued and Z is second countable, then for each $y_0 \in Y$, there is a dense G_δ subset D of X such that F is upper (resp. lower) semi-continuous at each point of $D \times \{y_0\}$.

1. Introduction

The notion of quasi-continuity is due to Volterra [1], who observed that every separately continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is quasi-continuous. Later on Kempisty [11] formulated the definition of quasi-continuity for real functions. Kempisty's ideas have been used in the investigation of the continuity points of functions of two variables which are quasi-continuous in one variable and continuous in the other one [2,3,12,14,18]. Among them there is the following.

Theorem 1.1. *([17, Theorem 1]) Let X be a Baire space, Y be first countable and Z be regular. If $f : X \times Y \to Z$ is a function such that all its x-sections f_x are continuous and all its y-sections f^y are quasi-continuous, then f is strongly quasi-continuous.*

In 1976, G. Gruenhage [9] introduced a class of topological spaces, called W-spaces, which contains the class of all first countable spaces. It is known that Piotrowski's result can be generalized to the case when Y is a W-space [15].

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In 1975, Popa [19] generalized the notion of quasi-continuity for set-valued functions. Since then, some authors investigated various types of continuity of two variable set-valued functions [5,7,8]. In particular, T. Neubrunn proved the following.

Theorem 1.2. *([16, Theorems 2 and 4]) Let X be a Baire space, Y second countable and Z normal (resp. regular). Let $F : X \times Y \to 2^Z$ be a function such that for each $(x, y) \in X \times Y$, F_x is upper (resp. lower) quasi-continuous and F^y is both lower and upper quasi-continuous. Then F is upper (resp. lower) quasi-continuous.*

In this paper, we define a notion for strong upper and lower quasi-continuity of a two variable set-valued function. We apply a topological game argument to give a partial extension of the above results. More precisely, we will show that if X is a Baire space, Y is a W-space and Z is a normal (resp. regular) T_1-space, then $F : X \times Y \to 2^Z$ is strongly upper (resp. lower) quasi-continuous provided that F_x is upper (resp. lower) semi-continuous and F^y is lower (resp. upper and lower) quasi-continuous.

We also give an example to show that upper and lower strong quasi-continuity of a set-valued function does not imply strong quasi-continuity. However, when Z is second countable, we will prove that for any function $F : X \times Y \to 2^Z$ which is both upper and lower strong quasi-continuous and $y_0 \in Y$, there is a residual subset D of X such that F is quasi-continuous at each point of $D \times \{y_0\}$.

Let $C_u(F)$ (resp. $C_l(F)$) denote the set of all upper (resp. lower) semi-continuous points of a set-valued function F. The following result is due to J. Evert and T. Lipski.

Theorem 1.3. *([6, Theorems 15 and 16]) Let X be a topological space and Z be a second countable space. Suppose that $F : X \to 2^Z$ is a non-empty compact valued upper (resp. lower) quasi-continuous function. Then $X \setminus C_u(F)$ (resp. $X \setminus C_l(F)$) is of the first category.*

We will give a partial extension of the above result by showing that if $F : X \times Y \to 2^Z$ is compact-valued strongly upper (resp. lower) quasi-continuous function and Z is second countable, then for every $y_0 \in Y$, we can find a residual subset D of X such that F is upper (resp. lower) semi-continuous at each point of $D \times \{y_0\}$. In particular, when X is a Baire space, D is a dense G_δ subset of X.

2. Preliminaries

Throughout this paper, we will assume that X, Y and Z are topological spaces. Let us start this section by introducing the following topological games.

The Banach–Mazur game $\mathcal{BM}(X)$ is a topological game played by two players α and β as follows. Player β starts a game by selecting a nonempty open set U_1 of X; then player α chooses a non-empty open set $V_1 \subset U_1$. When (U_i, V_i), $1 \leqslant i \leqslant n-1$, have been defined, player β picks a nonempty open set $U_n \subset V_{n-1}$ and α answers by selecting a nonempty open set $V_n \subset U_n$. The player α wins the play $(U_i, V_i)_{i \geqslant 1}$ if $(\bigcap_{n=1}^{\infty} V_n) \neq \emptyset$. Otherwise, the player β is said to have won the play.

By a strategy for one of the players, we mean a rule that specifies each move of the player. We say that the player α has a winning strategy for the game $\mathcal{BM}(X)$ if there exists a strategy s, such that α wins all plays provided that he/she acts according to the strategy s. In this case, we say that X is an α-favorable space. It is known that X is a Baire space if and only if the player β does not have a winning strategy in the game $\mathcal{BM}(X)$ (see e.g. [20, Theorems 1 and 2]). Therefore every α-favorable space X is a Baire space, however, the converse is not true in general. The interested reader is referred to [4,10,21,22] for further information.

The following game was introduced by G. Gruenhage [9] to define a class of topological spaces which strictly contains all first countable topological spaces.