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Coincidence of the Isbell and Scott topologies on domain function spaces [☆]

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ABSTRACT

Let $[X \to L]$ be the set of all continuous mappings from a topological space X to a domain L with the pointwise order. Let $Is[X \to L]$, $\sigma[X \to L]$ be the Isbell and Scott topologies on $[X \to L]$ respectively. In this paper, the question of when the Isbell and Scott topologies coincide on $[X \to L]$ is considered. Main results are:

(1) If L is a bicomplete domain, then (i) $Is[X \to L] = \sigma[X \to L]$ for all core compact spaces X if and only if L is bounded complete; (ii) $Is[X \to L] = \sigma[X \to L]$ for all core compact and compact spaces X if and only if L is conditionally bounded complete.
(2) Let L be a dcpo consisting of a least element and a decreasing sequence with two lower bounds, then $Is[X \to L] = \sigma[X \to L]$ for all core compact spaces X with a countable base.
(3) Let X be a well rooted Lawson compact domain and L an FL-domain, then $Is[X \to L] = \sigma[X \to L]$.

1. Introduction and preliminaries

Let X be a topological space and L a dcpo (i.e., L is a partially ordered set in which every directed set has the supremum) equipped with the Scott topology $\sigma(L)$. Write $[X \to L]$ for the set of all continuous mappings from X to L; then the set $[X \to L]$ with the pointwise order is again a dcpo. Lawson and Mislove [5] posed the following problem:

Problem. Let X be a topological space and L a dcpo equipped with the Scott topology. Under what conditions on L do the Isbell and Scott topologies on $[X \to L]$ agree?

This problem was considered by Lawson [6], Liang and Liu [7]. Recall that a dcpo L is an *L-dcpo* if for all $x \in L$ the set $\downarrow x = \{y \in L \mid y \leqslant x\}$ is a complete lattice. Liu and Liang obtained the following result.

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Theorem 1.1. *([7]) Let L be a continuous L-dcpo with a least element. Then the Isbell and Scott topologies on $[X \to L]$ agree for all core compact spaces X if and only if L is bounded complete.*

We show that it is not necessary to assume that L is an L-dcpo in the above theorem and the result can be improved. Recall that a dcpo is said to be *bicomplete* if each filtered subset of L has an infimum. We prove that if L is a bicomplete domain, then the Isbell and Scott topologies on $[X \to L]$ coincide for all core compact spaces X if and only if L is bounded complete. Furthermore, we consider the case of the domain L without least element. We show that if L is a bicomplete domain, the Isbell and Scott topologies on $[X \to L]$ are equal for all core compact and compact spaces X if and only if L is a domain in which all non-empty subsets having an upper bound have a least upper bound.

If the domain L is not bicomplete, we consider an example: Let L be a dcpo consisting of a least element and a decreasing sequence with two lower bounds. Obviously, L is not bicomplete. We show that the Isbell and Scott topologies on $[X \to L]$ agree for all core compact spaces X with a countable base. In particular, the two topologies on $[L \to L]$ agree. However, $[L \to L]$ is not continuous. So we obtain a noncontinuous dcpo, on which the Scott topology has Scott-open filters as a base. This is related to an open problem [5] posed by Lawson and Mislove: characterize those dcpos for which the open-set lattices with respect to the Scott open filter topology are continuous.

In the study of the maximal Cartesian closed full subcategories of domains without least element, Jung [4] introduced well-rooted domains and *FL*-domains. We show that if X is a well-rooted Lawson compact domain and L is an *FL*-domain, then the Isbell and Scott topologies on $[X \to L]$ are equal.

We quickly recall some basic notions concerning domains and function spaces (see [1–3,9]). Let L be a dcpo. For all $x, y \in L$, $x \ll y$ if and only if for every directed subset E of L satisfying $y \leqslant \bigvee E$, there exists $e \in E$ such that $x \leqslant e$. For any $x \in L$, write $\Downarrow x = \{y \in L \mid y \ll x\}$. A poset L is said to be continuous if for any $x \in L$, $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$. A continuous dcpo is also called a domain. A subset U of L is Scott open if U is an upper set and for any directed subset E, $\bigvee E \in U$ implies $U \cap E \neq \emptyset$. The set of all Scott open subsets $\sigma(L)$ is called the Scott topology. The Lawson topology of L is the topology obtained by taking $\{U \setminus \uparrow x \mid U \in \sigma(L), x \in L\}$ as a subbase for the open sets.

L is called *bounded complete* if L is a dcpo and each subset with an upper bound has a supremum; L is called *conditionally bounded complete* if L is a dcpo and each non-empty subset with an upper bound has a supremum. If a smallest element is added to a conditionally bounded complete dcpo without least element, the result is a bounded complete dcpo. A continuous L-dcpo is also called an *L-domain*. Obviously, a bounded complete domain is an L-domain, but the converse is not true.

A topological space X is said to be core compact if and only if its open set lattice $\Omega(X)$ is continuous. Even if X is not core compact, $\Omega(X)$ is a complete lattice, and so it is has its own intrinsic Scott topology.

Definition 1.2. Let X be a topological space and L a dcpo. The Isbell topology on $[X \to L]$, written $Is[X \to L]$, is the topology obtained by taking as a subbase for the open sets all sets of the form

$$N(H, V) = \{f \in [X \to L] \mid f^{-1}(V) \in H\},$$

where H is a Scott open set in $\Omega(X)$ and V is a Scott open set in L.

2. The function spaces for bounded complete domains

The following examples are useful for our discussion.

Example 2.1. Let L_1 be a two-element domain with the discrete order. Let $X = \mathbf{N}$ with the discrete topology. Note that L_1 and $\Omega(X)$ have countable bases. So does $Is[X \to L_1]$. However, $[X \to L_1]$ is an