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Ultrafilters on metric spaces

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ABSTRACT

Let X be an unbounded metric space, $B(x,r) = \{y \in X\colon\ d(x,y) \leqslant r\}$ for all $x \in X$ and $r \geqslant 0$. We endow X with the discrete topology and identify the Stone–Čech compactification βX of X with the set of all ultrafilters on X. Our aim is to reveal some features of algebra in βX similar to the algebra in the Stone–Čech compactification of a discrete semigroup [6].

We denote $X^\# = \{p \in \beta X\colon$ each $P \in p$ is unbounded in $X\}$ and, for $p, q \in X^\#$, write $p \parallel q$ if and only if there is $r \geqslant 0$ such that $B(Q,r) \in p$ for each $Q \in q$, where $B(Q,r) = \bigcup_{x\in Q} B(x,r)$. A subset $S \subseteq X^\#$ is called invariant if $p \in S$ and $q \parallel p$ imply $q \in S$. We characterize the minimal closed invariant subsets of X, the closure of the set $K(X^\#) = \bigcup\{M\colon\ M$ is a minimal closed invariant subset of $X^\#\}$, and find the number of all minimal closed invariant subsets of $X^\#$.

For a subset $Y \subseteq X$ and $p \in X^\#$, we denote $\triangle_p(Y) = Y^\# \cap \{q \in X^\#\colon\ p \parallel q\}$ and say that a subset $S \subseteq X^\#$ is an ultracompanion of Y if $S = \triangle_p(Y)$ for some $p \in X^\#$. We characterize large, thick, prethick, small, thin and asymptotically scattered spaces in terms of their ultracompanions.

1. Introduction

Let X be a discrete space, and let βX be the Stone–Čech compactification of X. We take the points of βX to be the ultrafilters on X, with the points of X identified with the principal ultrafilters, so $X^* = \beta X \setminus X$ is the set of all free ultrafilters. The topology of βX can be defined by stating that the set of the form $\overline{A} = \{p \in \beta X\colon\ A \in p\}$, where A is a subset of X, are base for the open sets. The universal property of βX states that every mapping $f : X \to Y$, where Y is a compact Hausdorff space, can be extended to the continuous mapping $f^\beta : \beta X \to Y$.

If S is a discrete semigroup, the semigroup multiplication has a natural extension to βS, see [6, Chapter 4]. The compact right topological semigroup βS has a plenty of applications to combinatorics, topological algebra and functional analysis, see [3,5,6,21,22].

Now let (X,d) be a metric space, $B(x,r) = \{y \in X\colon\ d(x,y) \leqslant r\}$ for all $x \in X$ and $r \geqslant 0$. A subset V of X is *bounded* if $V \subseteq B(x,r)$ for some $x \in X$ and $r \geqslant 0$. We suppose that X is *unbounded*, endow βX with the *discrete* topology and, for a subset Y of X, put

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$$Y^\# = \{p \in \beta X\colon \text{ each } P \in p \text{ is unbounded in } X\},$$

and note that $Y^\#$ is closed in βX.

For $p, q \in X^\#$, we write $p \parallel q$ if and only if there is $r \geqslant 0$ such that $B(Q, r) \in p$ for each $Q \in q$, where $B(Q, r) = \bigcup_{x \in Q} B(x, r)$. The parallel equivalence was introduced in [8] in more general context of balleans and used in [1,10–13].

For $p \in X^\#$, we denote $\bar{\bar{p}} = \{q \in X^\#\colon\ p \parallel q\}$ and say that a subset S of $X^\#$ is invariant if $\bar{\bar{p}} \subseteq S$ for each $p \in S$. Every nonempty closed invariant subset of $X^\#$ contains some non-empty minimal (by inclusion) closed invariant subset. We denote

$$K(X^\#) = \bigcup \{M\colon\ M \text{ is a minimal closed invariant subset of } X^\#\}.$$

After a short technical Section 2, we show in Section 3 how one can detect whether $S \subseteq X^\#$ is a minimal closed invariant subset, and whether $q \in X^\#$ belongs to the closure of $K(X^\#)$ in $X^\#$. We prove that the set of all minimal closed invariant subsets of $X^\#$ has cardinality $2^{2^{asden\, X}}$, where $asden\, X = \min\{|L|\colon\ L \subseteq X \text{ and } X = B(L, r) \text{ for some } r \geqslant 0\}$.

In Section 5 we show that from the ballean point of view the minimal closed invariant subsets are counterparts of the minimal left ideal in βG, where G is a discrete group. Thus, the results of Section 3 are parallel to Theorems 4.39, 4.40 and 6.30(1) from [6].

In Section 3 we use the following classification of subsets of a metric space. We say that a subset Y of X is

- *large* if $X = B(Y, r)$ for some $r \geqslant 0$;
- *thick* if, for every $r \geqslant 0$, there exists $y \in Y$ such that $B(y, r) \subseteq Y$;
- *prethick* if $B(Y, r)$ is thick for some $r \geqslant 0$;
- *small* if $X \setminus Y \cap L$ is large for every large subset L of X;
- *thin* if, for each $r \geqslant 0$, there exists a bounded subset V of X such that $B(y, r) \cap Y = \{y\}$ for each $y \in Y \setminus V$.

It should be mentioned that some of these notions have their counterparts in semigroups. Thus, large and prethick subsets correspond to syndedic and piecewise syndedic subsets [6, p. 101]. For definition of thick subset of a semigroup see [6, p. 104].

We note that Y is small if and only if Y is not prethick [15, Theorem 11.1], and the family of all small subsets of X is an ideal in the Boolean algebra of all subsets of X [15, Theorem 11.2]. Hence, for every finite partition of X, at least one cell is prethick.

For $p \in X^\#$ and $Y \subseteq X$, we put

$$\triangle_p(Y) = \bar{\bar{p}} \cap Y^\#,$$

and say that $\triangle_p(Y)$ is a *p-companion* of Y. A subset $S \subseteq X^\#$ is called an *ultracompanion* of Y if $S = \triangle_p(Y)$ for some $p \in X^\#$.

In Section 4 we characterize all above defined subsets of X and asymptotically scattered subsets from [14] in terms of their ultracompanions.

During the exposition, X is an *unbounded metric space.*

2. Parallelity

Lemma 2.1. *Let Y be a subset of X, $r \geqslant 0$, $q \in X^\#$. If $B(Y, r) \in q$ then there exists $s \in Y^\#$ such that $q \parallel s$.*