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Pseudocompact rectifiable spaces

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A topological space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. We firstly define the concept of rectifiable completion of rectifiable spaces and study some properties of rectifiable complete spaces, and then we mainly show that: (1) Each pseudocompact rectifiable space G is a Suslin space, which gives an affirmative answer to V.V. Uspenskij's question (Uspenskij, 1989 [29]); (2) Each pseudocompact infinite rectifiable space contains a non-closed countable set; (3) Each pseudocompact rectifiable space G is sequentially pseudocompact; (4) Each infinite pseudocompact rectifiable space with a continuous weak selection is homeomorphic to the Cantor set; (5) Each first-countable ω-narrow rectifiable space has a countable base. Moreover, some examples of rectifiable spaces are given and some questions concerning pseudocompactness on rectifiable spaces are posed.

1. Introduction

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product map of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A topological space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a *rectification* on G. It is well known that rectifiable spaces are a good generalization of topological groups. In fact, for a topological group with the neutral element e, it is easy to see that the map $\varphi(x, y) = (x, x^{-1}y)$ is a rectification on G. However, the 7-dimensional sphere S_7 is rectifiable but not a topological group [29, §3]. Further, it is easy to see that each rectifiable space is homogeneous. Recently, the study of rectifiable spaces has become an interesting topic in topological algebra, see [3,4,10,14–17].

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This paper is organized as follows: In Section 3, we mainly discuss the compact rectifiable spaces, and show that each infinite compact rectifiable space contains a homeomorphic copy of $D^{\omega(G)}$, and every compact α_4-rectifiable space is α_1; In Section 4, we mainly define the rectifiable complete of rectifiable spaces, and show that each locally compact rectifiable space is rectifiable complete; In Section 5, we mainly discuss the pseudocompact rectifiable spaces; In Section 6, we mainly discuss the precompact rectifiable spaces; In Section 7, some examples of rectifiable spaces are given and some questions concerning pseudocompactness on rectifiable spaces are posed.

2. Preliminaries

Definition 2.1. Let X be a topological space. For $i = 1, 4$ we say that X is an *α_i-space* if for each countable family $\{S_n\colon\ n \in \mathbb{N}\}$ of sequences converging to some point $x \in X$ there is a sequence S converging to x such that:

(α_1) $S_n \setminus S$ is finite for all $n \in \mathbb{N}$;

(α_4) $S_n \cap S \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.

Recall that a family $\mathcal{U}$ of non-empty open sets of a space X is called a *π-base* if for each non-empty open set V of X, there exists a $U \in \mathcal{U}$ such that $U \subset V$. The *π-character* of x in X is defined by $\pi\chi(x, X) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a local π-base at x in $X\}$. The *π-character of* X is defined by $\pi\chi(X) = \sup\{\pi\chi(x, X)\colon\ x \in X\}$. The *character* of x in X is defined by $\chi(x, X) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a local base at x in $X\}$. The *character of* X is defined by $\chi(X) = \sup\{\chi(x, X)\colon\ x \in X\}$. By $\omega(X)$ we denote the weight of X, the least cardinality of bases for X. *Compact-covering number* $k(X)$ is the least cardinal of a family of compact subsets of X whose union is X.

Let ω^ω denote the family of all functions from $\mathbb{N}$ into $\mathbb{N}$. For $f, g \in \omega^\omega$ we write $f <^* g$ if $f(n) < g(n)$ for all but finitely many $n \in \mathbb{N}$. A family $\mathscr{F}$ is *bounded* if there is a $g \in \omega^\omega$ such that $f <^* g$ for all $f \in \mathscr{F}$, and is *unbounded* otherwise. We denote by $\flat$ the smallest cardinality of an unbounded family in ω^ω. It is easy to see that $\omega < \flat \leqslant \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum.

A space X is *countably compact* if each countably open cover of X has a finite subcover. A space X is *locally compact* if every point of G has a compact neighborhood. A Tychonoff space X is *pseudocompact* if every real-valued continuous function on X is bounded.

Theorem 2.2. *([5,10,28]) A topological space G is rectifiable if and only if there exist $e \in G$ and two continuous maps $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x \in G$, $y \in G$ the next identities hold*:

$$p\big(x, q(x, y)\big) = q\big(x, p(x, y)\big) = y \quad \textit{and} \quad q(x, x) = e.$$

Given a rectification φ of the space G, we may obtain the mappings p and q in Theorem 2.2 as follows. Let $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$. Then the mappings p and q satisfy the identities in Theorem 2.2, and both are open mappings.

Let G be a rectifiable space, and let p be the multiplication on G. Further, we sometimes write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$; since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x. Hence a rectifiable space G is a topological algebraic system with operations p and q, a 0-ary operation e, and identities as above. It is easy to see that this algebraic system need not satisfy the associative law about the multiplication operation p. Clearly, every topological loop is rectifiable.

Let A be a subspace of a rectifiable space G. Then A is called *a rectifiable subspace of* G [17] if we have $p(A, A) \subset A$ and $q(A, A) \subset A$.