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The smooth structure of the moduli space of a weighted series-parallel graph

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ABSTRACT

Given a weighted graph (G, l) and its associated moduli space $M(G, l)$, then a sufficient condition is provided which ensures that $M(G, l)$ is a smooth manifold whenever G is a series-parallel graph.

1. Introduction

Given a polygonal graph G, i.e. a cycle, with specified lengths for the edges, then necessary and sufficient conditions for being able to *draw*, or realize, such a graph in the Euclidean plane with the preassigned edge lengths are given in [7]. Moreover, it is also shown in [7] (among other things) that, for a polygonal graph, the *space* of possible realizations, or *drawings*, in the plane is a smoothly embedded submanifold of $\mathbb{R}^d$ if there are no so-called straight line realizations of the graph with the specified edge lengths. Further results in a similar vein can be found in [2,8] and [10]. In the case where the graph belongs to a class of graphs known as *series-parallel* graphs (which contain polygonal graphs), necessary and sufficient conditions for being able to *draw* such a graph in the Euclidean plane with preassigned edge lengths are given in [1].

The main result of this note, namely Theorem 4.4, generalizes the result in [7] relating to the space of possible realizations in the plane. It states that for a series-parallel graph G with specified edge lengths, the space of possible realizations in the plane is a smoothly embedded submanifold of $\mathbb{R}^d$ if for any realization there is no polygonal subgraph of G all of whose vertices are realized as a straight line.

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Note that a graph G with specified edge lengths is sometimes termed a *linkage*, for instance, [1] and [6]. This is not standard terminology, however, it formalizes the intuition that a weighted graph is the mathematical model for a mechanical linkage consisting of hinges and bars that are constrained to move in a plane, where the issue of self-intersections is ignored.

2. Preliminaries

A *graph* G is a pair (V_G, E_G) where V_G, the *vertex set* of G, is finite, and E_G, the *edge set* of G, is a multiset whose elements are elements of $[V_G]^2$, the set of 2-element subsets of V_G. An edge $\{u, v\}$ is denoted uv in the sequel. In this note, graphs have neither parallel edges nor loops. Clearly this assumption does not impact on the generality of the overall result as multiple edges with the same endpoints would have to be assigned the same length, and all loops assigned the length zero, or else it would not be possible to *draw* G in the plane. For further detail regarding graph theory, see [3,5] or [11]. A *length function* on a graph G is a function $l : E_G \to \mathbb{R}^+$. In the sequel, if $H \subset G$, then $l|_H$ denotes the restriction of l to H. A *weighted graph* is a pair (G, l) where G is a graph and l is a length function on G. The *configuration space* $C(G, l)$ of a weighted graph (G, l) is defined as

$$C(G,l) = \bigl\{p : V_G \to \mathbb{E}^2 \bigm| d\bigl(p(u), p(v)\bigr) = l(uv) \text{ for all } uv \in E_G\bigr\}$$

where $d(x, y)$ is the standard Euclidean distance, noting that $C(G, l)$ is equipped with the topology it inherits as a subspace of $(\mathbb{R}^2)^{|V_G|}$.

Each $p \in C(G, l)$ is called a *realization* of (G, l). If there exists a realization of (G, l), then the weighted graph (G, l) is said to be *realizable*. Given a graph G with vertex set V_G then the group $\mathbb{E}^+(2)$ of *orientation preserving isometries* of $\mathbb{E}^2$ acts on $C(G, l)$ by $(g \cdot p)(v) = g \cdot (p(v))$ for all $v \in V_G$. Given a weighted graph (G, l) and the configuration space $C(G, l)$, then the *moduli space* $M(G, l)$ of (G, l) is the quotient space

$$M(G,l) = C(G,l)/\mathbb{E}^+(2).$$

Elements of a moduli space $M(G, l)$ are equivalence classes and so are usually denoted by $[p]$, however, whenever no confusion can arise, by a slight abuse of notation, the elements of $M(G, l)$ are simply denoted p in the sequel. A subspace of a configuration space which is utilized in the sequel is now described. Given a weighted graph (G, l) with vertices $u_0, v_0 \in V_G$ such that $u_0v_0 \in E_G$, then define

$$C_{u_0,v_0}(G,l) = \bigl\{p \in C(G,l) \bigm| p(u_0) = (0,0) \text{ and } p(v_0) = \bigl(l(u_0v_0), 0\bigr)\bigr\}.$$

Note that $C_{u_0,v_0}(G, l)$ and $C_{v_0,u_0}(G, l)$ are different as sets but are homeomorphic topological spaces. Observe that given a weighted graph (G, l) then the space $C_{u_0,v_0}(G, l)$ is homeomorphic to the moduli space $M(G, l)$. In the sequel, $M(G, l)$ is understood to refer to $C_{u_0,v_0}(G, l)$. In particular, $M(G, l)$ is a real algebraic subvariety of $\mathbb{R}^d$ where $d = 2|V_G| - 3$. The term *smooth* is taken to mean differentiable of class C^∞ and standard definitions relating to differentiability and smooth manifolds can be found in [9].

3. Series parallel graphs

A *two-terminal graph* is an ordered triple (G, a, c) where a and c are distinct vertices of G. The vertex a is termed the source and the vertex c is termed the sink. Collectively, the source and sink vertices are referred to as the *terminal vertices* of G. Two-terminal graphs are henceforth referred to as TTGs.

Let $G \cup_{x\sim y} H$ denote the union of two disjoint graphs G and H in which $x \in V_G$ is identified with $y \in V_H$. A *series composition* of two TTGs (G, a, c) and (H, b, d) is the TTG $(G \cup_{c\sim b} H, a, d)$ which is denoted $(G, a, c) \circ (H, b, d)$. The sink of (G, a, c) and the source of (H, b, d) are defined to be identified.