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Monotone normality and stratifiability from a pointfree point of view ☆

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A R T I C L E I N F O

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A B S T R A C T

Monotone normality is usually defined in the class of T_1 spaces. In this paper we study it under the weaker condition of subfitness, a separation condition that originates in pointfree topology. In particular, we extend some well known characterizations of these spaces to the subfit context (notably, their hereditary property and the preservation under surjective continuous closed maps) and present a similar study for stratifiable spaces, an important subclass of monotonically normal spaces. In the second part of the paper, we extend further these ideas to the lattice theoretic setting. In particular, we give the pointfree analogues of the previous results on monotonically normal spaces and introduce and investigate the natural pointfree counterpart of stratifiable spaces.

1. Introduction

Fifty years ago, Borges [4, Lemma 2.1] introduced and Zenor ([34], see [16]) named the notion of monotone normality, a strengthening of normality. Since that pioneering papers, there has been an extensive literature

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on the topic (see e.g. [5,7,25,32] for references). Every metrizable space and every linearly ordered space is monotonically normal. In fact, it could be argued that whenever a space can be shown "explicitly" to be normal, then it is probably monotonically normal.

Monotone normality is usually treated in the class of T_1 spaces (in this context, a space is monotonically normal iff it is hereditarily monotonically normal [5], i.e., every its subspace is monotonically normal). Apart [21] and, more recently, [22,12,13,15], monotone normality has been considered in the restricted class of T_1 spaces. In [12], Gutiérrez García, Mardones-Pérez and de Prada Vicente undertook the study of monotone normality free of the T_1 property and obtained new characterizations of monotone normality for general spaces. In addition, they showed that monotone normality is not, in general, a hereditary property.

In the present paper, by approaching the problem from a pointfree point of view, we are able at the same time to improve these results and to extend them to the pointfree setting. Our primary motivating question is the following: is there any separation axiom weaker than T_1 under which monotone normality becomes an hereditary condition?

The T_1 axiom for spaces is so heavily dependent on points that one cannot expect an exact pointfree counterpart for it. Subfit frames [19] and, sometimes, unordered (T_U) frames [19,20] have been considered as candidates but both fail to coincide with the T_1 property in the spatial case. The former form a strictly weaker counterpart of T_1 spaces [24]. They were introduced by Isbell in [19] and independently (as *conjunctivity*, because it is the opposite of the disjunctive property for distributive lattices) by Simmons [30]. A frame L is said to be *subfit* if

$$a \nleq b \quad \Longrightarrow \quad \exists c \in L\colon\ a \vee c = 1 \neq b \vee c. \tag{Sfit}$$

As remarked by Isbell [19] (and also by Simmons [30]), given a space $(X, \mathcal{O}X)$, the frame $\mathcal{O}X$ of open sets is subfit if and only if the underlying space satisfies the following condition:

$$\forall U \in \mathcal{O}X, \quad \forall x \in U, \quad \exists y \in \overline{\{x\}} \quad \text{such that} \quad \overline{\{y\}} \subseteq U. \tag{Conj}$$

Simmons (see e.g. [31, Lemma 4.8]) noted that

$$T_1 = (\text{Conj}) + T_D$$

where T_D is the familiar separation axiom between T_0 and T_1 (in fact, much closer to T_0 than to T_1) due to Aull and Thron [1], requiring that each point $x \in X$ has an open neighborhood U such that $U \smallsetminus \{x\}$ is also open.

Our main goal with this paper is to study the role of subfitness within monotone normality, first in spaces and then in the more general pointfree setting. The notion of a stratifiable frame will appear naturally as an interesting subclass of monotonically normal frames. They are the pointfree counterpart of the stratifiable spaces introduced by Ceder [6] and also studied by Borges [4] (to whom the name stratifiable is due). In particular, we will see that monotone normality is hereditary under subfitness while stratifiability is always hereditary. Further, we will study the preservation of both properties under closed maps.

The paper is organized as follows. In Section 2 we study the role of the subfitness axiom on monotonically normal spaces with the aim of extending the results in [5] from the class of T_1 spaces to the broader class of subfit spaces. In Section 3 we address perfectly normal spaces and stratifiable spaces. In Section 4 we show how those classical topological variants of normality can be naturally stated in a general lattice, yielding natural dual concepts closely related to that of extremal disconnectedness. These first sections emphasize the role (and usefulness) of the pointfree point of view in clarifying classical topological concepts and ideas and underlying principles. After recalling, in Section 5, the background on the category of frames and the corresponding pointfree approach to topology needed in the last two sections of the paper, we broaden the extent of the topological ideas of the first sections, by introducing and investigating monotonically normal frames (Section 6) and stratifiable frames (Section 7).