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Sequential properties of lexicographic products

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ABSTRACT

In this article, using the characterization of almost P-points of a linearly ordered topological space (LOTS) in terms of sequences, we observe that in the category of linearly ordered topological spaces, quasi F-spaces and almost P-spaces coincide. This coincidence gives examples of quasi F-spaces with no F-points. We also use the characterization of sequentially connected LOTS in terms of almost P-points to show that whenever each LOTS X_n has first and last elements, the lexicographic product $\prod_{n=1}^{\infty} X_n$ is sequentially connected if and only if each X_n is. Whenever each X_n is a LOTS without first and last elements, then it is shown that $\prod_{n=1}^{\infty} X_n$ is always a sequential space. The lexicographic product $\prod_{\alpha<\omega_1} X_\alpha$, where ω_1 is the first uncountable ordinal, is also investigated and it is shown that if each X_α contains at least two points, then $\prod_{\alpha<\omega_1} X_\alpha$ is always an almost P-space (a quasi F-space) but it is neither sequential nor sequentially connected. Using this lexicographic product, we give an example of a quasi F-space in which the set of F-points and the set of non-F-points are dense. Whenever each X_α, $\alpha < \omega_1$, does not have first and last elements, we show that the lexicographic product $\prod_{\alpha<\omega_1} X_\alpha$ is a P-space without isolated points.

1. Introduction

For each f in $C(X)$, the ring of all real-valued continuous functions on a completely regular Hausdorff (Tychonoff) space X, $Z(f) = \{x \in X\colon f(x) = 0\}$ is called a *zeroset* and $X \setminus Z(f)$ is called a *cozeroset*. A point x in a completely regular Hausdorff space X is said to be a *P-point* (an *almost P-point*) if every G_δ-set or every zeroset containing x is a neighborhood of x (has a nonempty interior) and X is called a *P-space* (an *almost P-space*) if every point of X is a P-point (an almost P-point), see [1,6,8] for more details and properties of these spaces. A point x in a completely regular Hausdorff space X is said to be an *F-point* if the ideal $O^x = \{f \in C(X)\colon x \in \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, Z(f)\}$ is prime, where βX is the Stone–Čech compactification of X. A space X is called an *F-space* if every point of βX is an F-point. It is well-known that a space X is an F-space if and only if every cozeroset is C^*-embedded in X, see Theorem 14.25 in [6]. A *quasi F-space*

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is a space in which every dense cozeroset is C^*-embedded, see [3] for more properties and characterizations of quasi F-spaces.

Throughout this paper, topological spaces under consideration are linearly ordered spaces and the reader is referred to [4] and [6] for undefined terms and notations.

A *linearly ordered topological space (LOTS)* is a triple $(X, \tau, <)$, where $<$ is a linear ordering of the set X and τ is the usual open interval topology defined by $<$. For every x in a LOTS X, x^+ (x^-) denotes the immediate successor (predecessor) of x, if it exists and the set of all points $y \in X$ satisfying $y > x$ ($y < x$) is denoted by $(x, \rightarrow)$ ($(\leftarrow, x)$). A subset S of a LOTS X is said to be *cofinal* if, for every $x \in X$, there exists $s \in S$ such that $s \geqslant x$. It is well-known that every LOTS is a normal Hausdorff space, see [4] and a LOTS is connected if and only if it is *Dedekind-complete* (i.e., every nonempty subset with an upper bound has a supremum or equivalently, every nonempty subset with a lower bound has an infimum) and it does not have consecutive elements, see [6, 3O]. Whenever $(W, \preccurlyeq)$ is a well-ordered set and for every $\alpha \in W$, $(X_\alpha, \leqslant_\alpha)$ is a linearly ordered set, then the lexicographic product of the family $\{(X_\alpha, \leqslant_\alpha)\colon\ \alpha \in W\}$ is the set of all points $x = (x_\alpha)_{\alpha\in W}$ with the order $<$ defined as follows: if $x = (x_\alpha)_{\alpha\in W}$ and $y = (y_\alpha)_{\alpha\in W}$ have $x \neq y$, let σ be the first element of the set $\{\alpha \in W\colon\ x_\alpha \neq y_\alpha\}$ in the ordering $\prec$ and then define $x < y$ provided $x_\sigma <_\sigma y_\sigma$. In what follows, we shall omit putting subscripts on the various orderings because context will make clear which ordering is meant. The lexicographic product will be denoted by $\prod_{\alpha\in W} X_\alpha$. In this paper we consider two familiar well-ordered sets as index sets: ω_1, the set of all countable ordinals and $\mathbb{N}$, the set of all natural numbers. We also assume that each factor space X_α in the lexicographic product $\prod_{\alpha\in W} X_\alpha$ contains at least two elements.

In a LOTS X, we call a point x a *P^+-point* if every G_δ-set containing x contains an interval $[x, y)$ for some $x < y \in X$. *P^--points* will be defined similarly. If we define $O^+(x) = \{f \in C(X)\colon\ [x, y) \subseteq Z(f),\ \text{for some } y \in X,\ x < y\}$ and similarly $O^-(x) = \{f \in C(X)\colon\ (y, x] \subseteq Z(f),\ \text{for some } y \in X,\ x > y\}$, then clearly x is a P^+-point (P^--point) if and only if $O^+(x) = M_x$ ($O^-(x) = M_x$), where $M_x = \{f \in C(X)\colon\ f(x) = 0\}$. A LOTS X is said to be *P^+-space* (*P^--space*) if every point of X is a P^+-point (P^--point) and it is clear that a LOTS X is a P-space if and only if it is both a P^+-space and a P^--space. The space of countable ordinals is an example of a P^+-space which is not P^--space, whence it is not a P-space, see [4] and [6] for more details of the space of ordinals.

A subset A of a topological space X is said to be *sequentially open* if whenever $\{x_n\}$ is a sequence in X that converges to a point in A, then $\{x_n\}$ is eventually in A. Similarly a subset B of X is called *sequentially closed* if whenever there is a sequence in B that converges to some point $x \in X$, then $x \in B$. We recall that a space X is a *sequential space* if every sequentially closed (open) subset of X is closed (open). It is well-known that a LOTS is sequential if and only if it is first countable, see [4] and [9]. A space X is said to be *sequentially connected* if X cannot be expressed as the union of two nonempty disjoint sequentially closed (open) sets.

Our aim in this article is to reveal the importance of almost P-points and their role in characterizing of some sequential properties of lexicographic products. In Section 2, we give some preliminary results and cite some facts from [2]. By the characterization of almost P-points of a LOTS in terms of sequences, we observe that quasi F-spaces and almost P-spaces coincide in the category of linearly ordered spaces. Using the characterization of a sequential LOTS and a sequentially connected LOTS in terms of almost P-points, we observe that a LOTS is sequentially connected if and only if it is connected without any almost P-points. In Sections 3 and 4 we study the lexicographic product $\prod_{\alpha\in W} X_\alpha$, where the well-ordered set W is either the set of natural numbers $\mathbb{N}$ or the set of countable ordinals ω_1. We observe that the existence of almost P-points of the lexicographic product $\prod_{n\in\mathbb{N}} X_n$ depends on first and last elements of the factor spaces. Using this lexicographic product, we give examples of P^+-spaces and P^--spaces in which the set of almost P-points is dense. Whenever the set of countable ordinals is considered as an index set, we will show that the lexicographic product is always an almost P-space and whenever each factor space does not have first and last elements, then this lexicographic product will be a P-space without isolated points. Sequential