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On the construction of domains of formal balls for uniform spaces ☆

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Dedicated to the memory of Professor Sergio Salbany

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ABSTRACT

In the spirit of the well-known constructions of Edalat and Heckmann for metric spaces, we endow the set of formal (closed) balls of a given uniform space with a structure of poset and prove several of its properties, which extend to the uniform framework the corresponding ones of metric spaces. In particular, to show under what conditions this poset is a dcpo we introduce and discuss a weak notion of uniform completeness. Some illustrative examples are also given.

1. Introduction

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Q}$ and $\mathbb{N}$ denote the set of all real numbers, the set of all nonnegative real numbers, the set of all rational numbers and the set of all positive integer numbers, respectively.

In their celebrated paper [2], Edalat and Heckmann established nice and direct links between the theory of (complete) metric spaces and domain theory by means of the notion of a formal ball.

Let us recall that the set of formal (closed) balls of a metric space (X,d) is simply the set $\mathbf{B}X := X \times \mathbb{R}$. Each element (x,r) of $\mathbf{B}X$ is called a formal ball.

Edalat and Heckmann showed that the pair $(\mathbf{B}X, \sqsubseteq)$ is a poset where

$$(x,r) \sqsubseteq (y,s) \quad \Leftrightarrow \quad d(x,y) \leqslant r - s,$$

for all $(x,r),(y,s) \in \mathbf{B}X$.

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In fact, they proved, among other, the following important results for a metric space (X,d) (see [2, Theorems 6 and 13, and Corollary 10]).

(A) $(\mathbf{B}X,\sqsubseteq)$ is a continuous poset.
(B) (X,τ_d) is homeomorphic to $\mathrm{Max}(\mathbf{B}X)$ when it is endowed with the restriction of the Scott topology of $(\mathbf{B}X,\sqsubseteq)$.
(C) (X,d) is separable if and only if $(\mathbf{B}X,\sqsubseteq)$ is an ω-continuous poset.
(D) (X,d) is complete if and only if $(\mathbf{B}X,\sqsubseteq)$ is a dcpo.

Note that from (A) and (D) it follows that (X,d) is complete if and only if $(\mathbf{B}X,\sqsubseteq)$ is a continuous domain.

Later on, Heckmann [7] improved result (B) showing that the Scott topology of $(\mathbf{B}X,\sqsubseteq)$ admits a compatible weightable quasi-metric Q such that (X,d) is isometric to $(\mathrm{Max}(\mathbf{B}X),Q|_{\mathrm{Max}(\mathbf{B}X)})$.

Edalat and Heckmann's approach, which is motivated in part by the work of Lawson on maximal point spaces [12], was continued and extended by several authors to ultrametric spaces, Banach spaces, hyperspaces, partial metric spaces, quasi-metric spaces, etc. (see e.g. [1,3,8,9,14–18,20]).

The purpose of this paper is to study the natural problem of constructing a suitable structure of poset when the formal balls are defined on a uniform space and then to generalize Edalat and Heckmann's constructions to the uniform setting. In fact, we shall obtain uniform versions of results (A)–(D) above. In particular, the uniform counterpart of (D) requires a weak notion of completeness which will be introduced here. Finally, the extension to our framework of Heckmann's quasi-metric construction will be also discussed. Our methods and techniques are inspired on the ones developed in [2].

2. Background

We start this section with several notions and facts on domain theory which will be useful later on. Our basic reference is [5].

A partially ordered set, or poset for short, is a (nonempty) set X equipped with a (partial) order $\sqsubseteq$. It will be denoted by $(X,\sqsubseteq)$ or simply by X if no confusion arises.

A subset D of a poset X is directed provided that it is nonempty and every finite subset of D has upper bound in D.

A poset X is said to be directed complete, and is called a dcpo, if every directed subset of X has a least upper bound.

An element x of X is said to be maximal if the condition $x\sqsubseteq y$ implies $x=y$. The set of all maximal points of X will be denoted by $\mathrm{Max}((X,\sqsubseteq))$, or simply by $\mathrm{Max}(X)$ if no confusion arises.

Let X be a poset and $x,y\in X$; we say that x is way below y, in symbols $x\ll y$, if for each directed subset D of X having least upper bound z, the relation $y\sqsubseteq z$ implies the existence of some $u\in D$ with $x\sqsubseteq u$.

A poset X is continuous if it has a basis B, where B is said to be a basis for X if for all $x\in X$, the set $\{b\in B\colon\ b\ll x\}$ is directed with least upper bound x.

A continuous poset which is also a dcpo is called a continuous domain or, simply, a domain.

A continuous poset having a countable basis is said to be an ω-continuous poset.

The Scott topology $\sigma(X)$ of a continuous poset $(X,\sqsubseteq)$ is the topology that has as a base the collection of sets $\{y\in X\colon\ x\ll y\}$, $x\in X$.

If D is a subset of X, we denote by $\sigma(X)_{|D}$ the restriction of $\sigma(X)$ to D.

Now we recall the notion of a uniform structure and of a uniform space as introduced by Gillman and Jerison [6, Chapter 15].