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What are effective descent morphisms of Priestley spaces?

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ABSTRACT

We discuss the problem formulated in the title. We solve it only in two very special cases: for maps with finite codomains and for maps that are open and order-open, or, equivalently, open and order-closed.

0. Introduction

A morphism $p : E \to B$ in a category $\mathbb{C}$ with pullbacks is said to be an effective descent morphism if the pullback functor $p^* : (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ is monadic. This definition was used many times since the early nineties by various authors, who also explained where it comes from and how to deal with it. Nevertheless let us briefly recall:

- Intuitively, when $p : E \to B$ is a *good surjection*, one can think of $E = (E, p)$ as an extension of B. If so, then given a problem on a certain category $\mathbb{A}^B$ associated with B, one can try first to solve it for $\mathbb{A}^E$ and then to use descent from E to B. This requires to have an induced functor $p^* : \mathbb{A}^B \to \mathbb{A}^E$,

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and to be able to describe the category $\mathbb{A}^B$ as the category $Des(p)$, called the *category of descent data* for p, and constructed as the category of objects in $\mathbb{A}^E$ equipped with a certain additional structure defined using p^*. Accordingly, the morphism p is said to be an *effective descent morphism* if a certain comparison functor $\mathbb{A}^B \to Des(p)$ is a category equivalence. This general idea of descent theory is due to A. Grothendieck (see e.g. [3] and [4]).

- There are several ways, later proposed by several authors, to describe the category $Des(p)$ at various levels of generality recalled in the survey papers [8] and [7]. In the 'basic' case of *global descent*, which we are considering in the present paper: $\mathbb{A}^B = (\mathbb{C} \downarrow B)$ is the category of pairs (A, f), where $f : A \to B$ is a morphism in $\mathbb{C}$; the functor $p^* : (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ is defined by $p^*(A, f) = (E \times_B A, \pi_1)$ using the pullback $E \times_B A$ of p and f; p^* has a left adjoint $p_!$, which is defined by $p_!(D, g) = (D, pg)$, and $Des(p)$ is defined as the category $(\mathbb{C} \downarrow E)^{\mathbb{T}^p}$ of algebras over the corresponding monad $\mathbb{T}^p$ on $(\mathbb{C} \downarrow E)$; the monadicity of p^* means that the standard comparison functor $(\mathbb{C} \downarrow B) \to Des(p)$ is a category equivalence.
- The expression "good surjection" used above is suggested by the fact that when $\mathbb{C}$ is 'good' (e.g. Barr exact), p is an effective descent morphism if and only if it is a regular epimorphism. A general characterization of effective descent morphisms is given in [7], but there are many concrete examples, including $\mathbb{C} = \mathsf{Top}$ (the category of topological spaces), where a lot of further work is needed to understand its meaning. Some of them are mentioned in Example 0.1 below.

Example 0.1.

(a) For $\mathbb{C} = \mathsf{Top}$, the effective descent morphisms are characterized by J. Reiterman and W. Tholen [13] in terms of ultrafilter convergence.

(b) Let $\mathbb{C}$ be either the category of preorders (that is, sets equipped with a reflexive and transitive relation) or the category of finite preorders. Then $p : E \to B$ is an effective descent morphism if and only if for every $b_2 \leqslant b_1 \leqslant b_0$ in B there exists $e_2 \leqslant e_1 \leqslant e_0$ in E with $p(e_i) = b_i$ $(i = 0, 1, 2)$. This was shown in [5] (published as a preprint in 1999), as a simplified version of the above-mentioned Reiterman–Tholen result. Note that this result on preorders easily implies similar results for equivalence relations and for order relations (finite or not).

(c) Since the category of compact Hausdorff spaces is Barr exact and its regular epimorphisms are nothing but (continuous) surjections, its effective descent morphisms also are nothing but surjections. However, the same is true for Stone spaces, whose category is only regular; this was first observed by M. Makkai (unpublished).

(d) As explained in [2], using (b) and (c) one can easily describe effective descent morphisms of preordered Stone spaces and of ordered Stone spaces: they are the same as continuous maps that are effective descent morphisms of underlying preorders.

(e) For $\mathbb{C}$ being the category of compact (not necessarily Hausdorff) 0-dimensional spaces the effective descent morphisms are characterized in [6], although that category does not admit arbitrary pullbacks, and so the existence of relevant pullbacks becomes a part of the definition of an effective descent morphism there.

(f) Generalizing (c), all categories monadic over the category Set of sets (which includes all varieties of universal algebras) are Barr exact and their effective descent morphisms are exactly those morphisms that are mapped to surjections by the forgetful functor to Set. However, this is not the case for some quasi-varieties; the first simple counter-examples were given in the first part of [8], and much more information, also about relational structures was obtained by A.H. Roque (see [14–16]).

(g) When $\mathbb{C}$ is the opposite category of commutative rings (with 1), $p : E \to B$ is an effective descent morphism if and only if, considered as a B-module homomorphism, it is a pure monomorphism. We refer to the third part of [8] for the proof; however, that proof essentially follows the first published