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Interior and neighbourhood

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A B S T R A C T

The notion of interior operator is intimately related to that of neighbourhood operator. This relationship is presented here in the form of an adjunction or Galois connection, in the sense that on an object (a space) X, the neighbourhood operator ν_X admits a right adjoint which agrees with its associated interior operator i_X on the subobjects (subspaces). We establish this one-to-one correspondence and present some consequences of this relation.

1. Introduction

A subset V of a space X is a neighbourhood of a point $x \in X$ if and only if x is contained in the interior of V. This definition of a neighbourhood is at the very core of the adjunction that exists between interior and neighbourhood. Indeed, the above definition can be stated as follows:

$$\uparrow V \subseteq \mathcal{V}(x) \quad \text{if and only if} \quad \{x\} \subseteq Int(V),$$

where $\uparrow V$ is the principal filter generated by V in the power set of X ordered by inclusion, $\mathcal{V}(x)$ the set of neighbourhoods of x and $Int(V)$ the interior of V.

Categorical interior operators were introduced by Vorster in [17] and subsequently studied in [1,2] and [12]. They have recently been applied to the study of fuzzy spaces [11] and texture spaces [4]. On the other hand, neighbourhoods with respect to closure operators were introduced in [8] to study convergence

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on categories. The article [10] starts with the notion of neighbourhood operator as a primitive concept and proceeds to investigate convergence and compactness with respect to neighbourhoods. It is observed in [10,14] that interior operators correspond to a special class of neighbourhood operators, called *regular neighbourhood operators*. Apart from their greater generality, in some instances considering neighbourhood operators, instead of interior operators greatly facilitates investigations of topological properties (cf. [14]).

In this paper, we present both neighbourhood operators – usually denoted by ν – and interior operators as lax natural transformations such that on each object X we have an adjunction $\nu_X \dashv j_X$. Here j_X is (an extension of) the interior operator on X. This adjunction shortens dramatically proofs involving these operators, as the manipulation of interior and neighbourhood reduces merely to lax diagrams and the composition of adjunctions. A sample of such "calculus" can be observed in Section 3 and Section 4, where the usual methods [10,14,15] would be fairly long. On the other hand, this adjunction offers new perspectives on spaces by looking at topologies as suitable adjunctions. The latter consideration is more in the line of the article [9].

2. Preliminaries

We recall and present in this section a few results about partially ordered sets (posets).

Let us fix a poset P. The set of all up-closed sets on P, which shall be denoted by $P^{\uparrow}$, is endowed with the reverse inclusion $\preccurlyeq$ of subsets. There is a canonical injective monotone map $\uparrow : P \to P^{\uparrow}$ that takes each element $x \in P$ to its *up-closure* $\uparrow x = \{y \mid x \leqslant y\}$. The poset P is complete precisely when $\uparrow$ admits a right adjoint given by the infimum, $\inf : P^{\uparrow} \to P$. For the remainder of this section, P is assumed to be complete.

Definition 2.1. An *interior operation* i on P is given by a monotone map $i : P \to P$ such that $i \leqslant 1_P$, i.e., $i(x) \leqslant x$ for all $x \in P$.

Definition 2.2. A *neighbourhood operation* ν on P is given by a monotone map $\nu : P \to P^{\uparrow}$ such that $\uparrow \preccurlyeq \nu$, i.e., $\nu(x) \subseteq \uparrow x$ for all $x \in P$.

For a neighbourhood operation ν on P, we are interested in the following property which is satisfied by $\uparrow$: given $Q \subseteq P$, if $x \in \nu(q)$ for each $q \in Q$ then $x \in \nu(\sup Q)$. The operation ν has this property precisely when it admits a right adjoint.

Definition 2.3. A neighbourhood operation ν on P is a called a *left-adjoint neighbourhood operation* if it admits a right adjoint denoted by $\nu_* : P^{\uparrow} \to P$.

Lemma 2.4. *If ν is a left-adjoint neighbourhood operation on P, then the composition $\nu_* \uparrow$ is an interior operation on P.*

Now, since $P^{\uparrow}$ is always complete, the embedding $\uparrow$ admits a right Kan extension (cf. [13]) $Ran_i(\uparrow)$ along any interior operation i:

$$\begin{tikzcd} P \arrow[dr, "Ran_i(\uparrow)"'] & & P \arrow[ll, "i"'] \arrow[dl, "\uparrow"] \\ & P^{\uparrow} \arrow[u, phantom, "\preccurlyeq"] & \end{tikzcd}$$

Lemma 2.5. *The right Kan extension $Ran_i(\uparrow)$ is a neighbourhood operation on P for every interior operation i.*