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A model for function spaces

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ABSTRACT

Let $f : X \to Y$ be a map between simply connected spaces and $\bar{f} : \mathbb{L}(V) \to \mathbb{L}(W)$ its Quillen model. If the differential on $\mathbb{L}(V)$ has only linear and quadratic parts, we show that there is a Lie algebra structure on $\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W))$, making of it a Lie model of the function space $\mathrm{map}(X, Y; f)$.

1. Introduction

Throughout the paper spaces are assumed to be of the homotopy type of simply connected CW-complexes. We make use of Quillen models in rational homotopy theory for which we refer to [4,8] for details. To each space X, Quillen associates, in a functorial way, a differential graded algebra $\lambda(X)$ which characterizes its rational homotopy type. A *Lie model* of X is any differential graded Lie algebra of the homotopy type of $\lambda(X)$. Among these, a Quillen model of X is a Lie model of the form $(\mathbb{L}(V), \delta)$ in which $\mathbb{L}(V)$ denotes the Lie algebra generated by the graded vector space V.

Let $f : X \to Y$ be a map, in which X is finite and Y is of finite type, and denote by $\bar{f} : (\mathbb{L}(V), \delta) \to (\mathbb{L}(W), \delta')$ a Quillen model of f. The adjoint action of $T(W)$ on $\mathbb{L}(W)$ and the map $U\bar{f} : (TV, d) \to (TW, d')$ induces a TV-differential module structure on $\mathbb{L}(W)$. In [5], it was shown that

$$\pi_n\big(\Omega\,\mathrm{map}(X, Y; f)\big) \otimes \mathbb{Q} \cong Ext_n^{TV}\big(\mathbb{Q}, \mathbb{L}(W)\big). \tag{1}$$

However the right term is computed using a semifree resolution $(TV \otimes (\mathbb{Q} \oplus sV), D) \xrightarrow{\simeq} \mathbb{Q}$, where the differential is defined by

$$D(v \otimes 1) = dv \otimes 1, \qquad Dsv = v \otimes 1 - S(dv \otimes 1),$$

and S is the $\mathbb{Q}$-graded vector space map (of degree 1) defined by

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$$S(v \otimes 1) = 1 \otimes sv, \qquad S\big(1 \otimes (\mathbb{Q} \oplus sV)\big) = 0,$$
$$S(ax \otimes 1) = (-1)^{|a|} aS(x \otimes 1), \quad \forall a \in TV,\ |x| > 0 \quad [1,6].$$

Let $P = TV \otimes (\mathbb{Q} \oplus sV)$. Denote sv by $\bar{v}$ and $x \otimes y \in P \otimes P$ by $x|y$. If $(TV, d) = (TV, d_1 + d_2)$ where $d_1 v \in V$ and $d_2 v = \sum a_{ij} v_i v_j$, we define a TV-map $\Delta : P \to P \otimes_{TV} P$ by setting $\Delta(1) = 1|1$, $\Delta(\bar{v}) = \bar{v}|1 + 1|\bar{v} + \sum (-1)^{|v_i|} a_{ij}(\bar{v}_i|\bar{v}_j)$ and extend Δ to $TV \otimes (\mathbb{Q} \oplus sV)$ as a morphism of TV-modules. We define

$$S \otimes S : V \otimes V \to sV \otimes sV \subset P \otimes P$$

by

$$(S \otimes S)\Big(\sum_i v_i \otimes w_i\Big) = \sum_i (-1)^{|v_i|} (\bar{v}_i|\bar{w}_i). \tag{2}$$

Hence $\Delta(\bar{v}) = \bar{v}|1 + 1|\bar{v} + (S \otimes S)(d_2 v)$.

Following [2,7], we define a bracket on $\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W))$ using the composition

$$P \xrightarrow{\Delta} P \otimes_{TV} P \xrightarrow{\alpha \otimes \beta} \mathbb{L}(W) \otimes_{TV} \mathbb{L}(W) \xrightarrow{[-,-]} \mathbb{L}(W),$$

that is, $[\alpha, \beta] = [\,,\,] \circ (\alpha \otimes \beta) \circ \Delta$. Moreover the usual differential on $\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W))$ is defined by $\tilde{D}g = \delta' g - (-)^{|g|} gD$. Endowed with this bracket, $(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)), \tilde{D})$ becomes a differential graded Lie algebra.

Given a map $f : X \to Y$, the connected component of $\mathrm{map}(X, Y)$ containing f is denoted by $\mathrm{map}(X, Y; f)$. In this paper, we show the following result.

Theorem 1. *Under the bracket defined above,* $(\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)), \tilde{D})$ *is a Lie model of* $\mathrm{map}(X, Y; f)$.

2. A model for function spaces

A model for function spaces is described in [2] and the same authors study L_∞-models of based map spaces [3]. We recall here a model of $\mathrm{map}(X, Y; f)$ (see [2]).

For a differential graded Lie algebra (L, δ), the Cartan–Eilenberg construction leads to a cocommutative differential coalgebra $C_*(L) = (\bigwedge(sL), d_1 + d_2)$ where

$$d_1(sx_1 \wedge \cdots \wedge sx_k) = -\sum_{i=1}^{k} (-1)^{n_i} sx_1 \wedge \cdots \wedge s\delta x_i \wedge \cdots \wedge sx_k, \tag{3}$$
$$d_2(sx_1 \wedge \cdots \wedge sx_k) = \sum_{1 \leqslant i < j \leqslant k} (-1)^{n_{ij}} (-1)^{|sx_i|} s[x_i, x_j] \wedge sx_1 \wedge \cdots \widehat{sx_i} \cdots \widehat{sx_j} \cdots \wedge sx_k, \tag{4}$$

where $n_i = \sum_{j<i} |sx_j|$, n_{ij} is the Koszul sign of the permutation

$$(sx_1, \ldots, sx_k) \to (sx_i, sx_j, sx_1, \ldots, \widehat{sx_i}, \ldots, \widehat{sx_j}, \ldots, sx_k),$$

and the symbol ^ means deleted.

Let $\bar{f} : (L, \delta) \to (L', \delta')$ be a Lie model of f. We consider the composition map $\tilde{f}$ (of degree -1)

$$C_*(L) \xrightarrow{C_*(\bar{f})} C_*(L') \xrightarrow{p} sL' \xrightarrow{\cong} L'.$$