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On monotone paracompactness

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ABSTRACT

Gartside and Moody proved that a space is protometrizable if and only if it has a monotone star-refinement operator on open covers. They called this property monotone paracompactness but noted that it might be better termed monotone full-normality, and posed the problem of characterizing spaces with a monotone locally-finite operator on open covers. Stares studied related monotone properties but left the above problem open. We introduce Nötherianly locally-finite bases, show that protometrizable spaces have such bases, and that spaces with such bases have a monotone locally-finite operator and are monotonically normal. An example of a non-protometrizable LOTS due to Fuller is shown to have a Nötherianly locally-finite base. The product $L(\omega_1) \times (\omega + 1)$ though not hereditarily normal, has a monotone locally-finite operator, while $M \times (\omega + 1)$ (where M is the Michael line) does not.

0. Introduction

Introduced by Nyikos in [11], protometrizable spaces are a common generalization of metrizable spaces and non-Archimedean spaces. A space X is said to be *protometrizable* if it is paracompact and has an orthobase. Recall that a base $\mathcal{B}$ is an *orthobase* if whenever $\mathcal{B}' \subseteq \mathcal{B}$, either $\bigcap \mathcal{B}'$ is open, or $\mathcal{B}'$ is a local

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base for any point in $\bigcap \mathcal{B}'$. Gartside and Moody [6, Theorem 1] showed that a space X is protometrizable if and only if X is *monotonically paracompact*: X possesses a function $r : \mathcal{C} \to \mathcal{C}$ ($\mathcal{C}$ is the set of open covers of X) such that (1) for every $\mathcal{U} \in \mathcal{C}$, $r(\mathcal{U})$ star-refines $\mathcal{U}$, and (2) if $\mathcal{U}, \mathcal{V} \in \mathcal{C}$ and $\mathcal{U}$ refines $\mathcal{V}$ (denoted by $\mathcal{U} \prec \mathcal{V}$), then $r(\mathcal{U})$ refines $r(\mathcal{V})$. Gartside and Moody noted that their definition might be better termed *monotonically fully-normal*, and posed the problem of characterizing those spaces that possess a monotone locally-finite operator on open covers of X, that is,

Problem 0.1. ([6, Question 3]) Characterize those spaces X for which there is a function $r : \mathcal{C} \to \mathcal{C}$, which we call a monotone locally-finite operator, such that

(1) for every $\mathcal{U} \in \mathcal{C}$, $r(\mathcal{U})$ is a locally-finite refinement of $\mathcal{U}$, and
(2) if $\mathcal{U}, \mathcal{V} \in \mathcal{C}$ and $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U})$ refines $r(\mathcal{V})$.

Gruenhage briefly discusses this problem in [9], as well as Good, Marsh, McCluskey, McMaster in [7]. It appears the only progress is by Stares [17] who showed that different characterizations of paracompactness, when "monotonized", may result in different classes of spaces, and remarked that it is unknown whether the class of protometrizable spaces coincides with the class of spaces with a monotone locally-finite operator. Several authors studied recently the weaker but closely related properties of monotonically (countably) metacompact [1,3,13,15].

In this note we introduce Nötherianly locally-finite bases and show that protometrizable spaces have such bases, and in turn spaces with such bases have a monotone locally-finite operator. We also find examples distinguishing between these new classes of spaces, and pose related questions.

1. Nötherianly locally-finite bases

Nyikos [11] provided a characterization of protometrizable spaces through the scattering process. If $\mathcal{K}$ is a class of topological spaces, then we define $S(\mathcal{K})$ to be the class of spaces which are obtained by the *scattering process*: take any space in $\mathcal{K}$, isolate all the points of some subset, replace each such point by a space in $\mathcal{K}$, and repeat transfinitely, taking some subspace of the inverse limit space at limit ordinals. If $\mathcal{M}$ is the class of metrizable spaces, then Nyikos showed that the class of protometrizable spaces coincides with the class $S(\mathcal{M})$.

We first show that metric spaces possess a monotone locally-finite operator. The Arhangel'skii Metrization Theorem [4, Theorem 5.4.6] states that a topological space is metrizable if and only if it is a T_1-space and has a regular base. A base $\mathcal{B}$ for a topological space X is *regular* if for every point $x \in X$ and any neighborhood U of x there is a neighborhood $V \subset U$ of x such that the set of all members of $\mathcal{B}$ that meet both V and $X \setminus U$ is finite. Clearly any base $\mathcal{B}'$ contained in a regular base $\mathcal{B}$ must itself be regular. An element B of (any family of sets) $\mathcal{B}$ is called *maximal* (with respect to inclusion), if $B = B'$ whenever $B \subseteq B' \in \mathcal{B}$. If $\mathcal{B}$ is a regular base for a space X, then the family of its maximal elements $\mathcal{B}^m = \{B \in \mathcal{B}\colon\ B \text{ is maximal}\}$ is a locally-finite cover of X [4, Lemma 5.4.3].

Theorem 1.1. *Every metric space possesses a monotone locally-finite operator.*

Proof. Let $\mathcal{B}$ be a regular base for X. For any open cover $\mathcal{U}$, let $\mathcal{B}_{\mathcal{U}} = \{B \in \mathcal{B}\colon\ B \subseteq U \text{ for some } U \in \mathcal{U}\}$. It is easily verified that $r(\mathcal{U}) = \mathcal{B}^m_{\mathcal{U}}$ is the required monotone locally-finite operator. $\square$

As seen in the above proof, the "maximal" operator provides the needed monotonicity for a locally-finite operator. It suggests the following definition.