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Topological conjugacy, transitivity, and patterns

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ABSTRACT

Let f denote a continuous map of the compact interval to itself. Suppose that f is topologically transitive. We show that if f exhibits a pattern and has the same topological entropy as the pattern, then f is topologically conjugate to the linearization of the pattern.

1. Introduction and preliminaries

A basic problem in discrete dynamical systems is to determine when two maps are topologically conjugate. A useful invariant in doing this is the topological entropy; if two maps are topologically conjugate, then they must have the same topological entropy. It is natural to ask in various situations if the reverse implication also holds, that is can topological entropy be used to give a sufficient condition for topological conjugacy. In this paper, we obtain this type of result in the setting of continuous maps of the interval to itself. Our result involves finite invariant sets and combinatorial patterns.

Following [10] we use the term **pattern** to denote a map

$$\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}.$$

In the case that π cyclically permutes the elements of $\{1, 2, \dots, n\}$, we call π a **periodic orbit pattern** with period n. Also, for finite subsets of the real line, we will use the notation $\{x_1 < x_2 < \dots < x_n\}$ to denote the set $\{x_1, x_2, \dots, x_n\}$ with the ordering $x_1 < x_2 < \dots < x_n$.

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Let I denote a compact nondegenerate interval on the real line, and let $f : I \to I$. We say that f **exhibits the pattern** π if and only if there is a set $Q = \{y_1 < y_2 < \cdots < y_n\} \subseteq I$ with $f(y_i) = y_j$ when $\pi(i) = j$. In this case we may also say that f **exhibits the pattern** π **on the set** Q.

Observe that if $f : I \to I$ and Q is a finite subset of I with $f(Q) \subseteq Q$, then there exists a unique pattern π such that f exhibits the pattern π on the set Q. We call π **the pattern exhibited by** f **on** Q.

Let π be a pattern. There is an associated map $L_\pi : [1, n] \to [1, n]$ that agrees with π on the set $\{1, 2, \dots, n\}$ and is linear between each two consecutive points of $\{1, 2, \dots, n\}$. We call this map the **linearization of the pattern** π.

We will use the notation $h(f)$ to denote the topological entropy of the map f. A definition and basic results about topological entropy may be found in [3], [2], or [1].

If a continuous map f of the interval to itself exhibits a pattern π it is natural to look for sufficient conditions for f to be topologically conjugate to L_π. In particular, we might ask whether or not the necessary condition $h(f) = h(L_\pi)$ is also sufficient. Well-known examples show that this is not the case. One example is as follows. Consider the family of maps of $[0, 1]$ (known as the quadratic family) given by $f_a(x) = ax(1-x)$, where the parameter a runs from 0 to 4. Let π be the cyclic permutation which maps 1 to 2, 2 to 3, and 3 to 1. It is well-known (see for example Sections II.5, II.6, and III.1 of [8]) that there exists a nondegenerate interval of parameters, such that the map f_a exhibits the pattern π and has the same topological entropy as the map L_π, but is not topologically conjugate to L_π. In this paper, we show that with the additional assumption of topological transitivity we do obtain a sufficient condition. We prove the following result:

Theorem 1.1. *Let π be a pattern, and let L be the linearization of the pattern. Suppose $f : I \to I$ is continuous and f exhibits the pattern π. Suppose that f is topologically transitive, and $h(f) = h(L)$. Then f is topologically conjugate to L.*

Along the way to proving this result, we obtain some other results giving strict inequalities for the topological entropy of topologically transitive maps of the interval.

Note that our theorem is related to the following well-known result.

Theorem 1.2. *([5]) Let $f : I \to I$ be continuous. Then*

$$h(f) = \sup h(L_\pi),$$

the supremum taken over all periodic orbit patterns π exhibited by f.

Our theorem shows that for a topologically transitive, continuous map f, the supremum in Theorem 1.2 is actually a maximum if and only if f is topologically conjugate to L_π, for some periodic orbit pattern π exhibited by f.

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2. Matrices and graphs

We will need several results dealing with nonnegative matrices. We begin with the following well known result. A proof may be found in [6].

Theorem 2.1 *(Perron–Frobenius). Let A be a square matrix with entries non-negative real numbers. There exists an eigenvalue λ of A such that*

1. *λ is real and $\lambda \geqslant 0$;*
2. *for every eigenvalue μ of A, $|\mu| \leqslant \lambda$.*

Furthermore, $\lambda = \lim_{n\to\infty} |A^n|^{\frac{1}{n}} = \overline{\lim}_{n\to\infty}(Trace(A^n))^{\frac{1}{n}}$, where $|A| = \sum a_{i,j}$ and $Trace(A) = \sum a_{i,i}$.