Topology and its Applications 166 (2014) 10–14

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On the end of the cone metric spaces

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A R T I C L E   I N F O

Article history:
Received 6 May 2013
Received in revised form 13 February 2014
Accepted 13 February 2014

MSC:
54E35

Keywords:
Cone metric space
Ordered vector spaces
Order unit
Interior of cone
Metrizability

A B S T R A C T

Starting with the initial paper of Huang and Zhang [2] in 2007, more than six hundred papers dealing with cone metric spaces have been published so far. In this short note we present a different proof of the known fact that the notion of a cone metric space is not more general than that of a metric space.

1. Cone metric space

The aim of this short note is to reprove the main result of [4]. For the standard terminology of ordered vector spaces, we will use [1]. Taking into account [3], we give the following definition.

Definition 1.1. Let E be an ordered vector space. An element $e \in E$ is called an ***order unit*** if for each $x \in E$ there exists a $\lambda \in \mathbb{R}$ such that $x \leqslant \lambda e$. The set of order units of E will be denoted by $\operatorname{ou}(E)$.

We note that

$$\operatorname{ou}(E) + \operatorname{ou}(E) \subset \operatorname{ou}(E) \quad \text{and} \quad (0, \infty) \operatorname{ou}(E) \subset \operatorname{ou}(E).$$

The following definition is given in [2].

Definition 1.2. Let X be a non-empty set and E be an ordered Banach space with $\operatorname{int}(K) \neq \emptyset$ and K closed. A function $d : X \times X \to E$ is called a ***cone metric*** if:

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http://dx.doi.org/10.1016/j.topol.2014.02.004

(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \leqslant d(x,z) + d(y,z)$ for all $x, y, z \in X$.

In this case (X, E, K, d) is called a ***cone metric space***.

We write $0 \ll e$ when $e \in \text{int}(K)$. We can modify the cone metric space by replacing ordered Banach space by ordered topological vector space.

An ordered vector space E is called ***Archimedean*** if $x = 0$ whenever $\mathbb{Z}x \leqslant y$ in E; further, E is called ***almost Archimedean*** if $x, y \in E$ and $-\varepsilon x \leqslant y \leqslant \varepsilon x$ for all $\varepsilon > 0$, then $x = 0$. Obviously, every Archimedean ordered vector space is almost Archimedean, but not vice versa. One should stress that the property of being almost Archimedean is not restricted to in the theory of ordered vector spaces: most of the work on cone metric spaces is being done for the so-called *normal cone metric spaces*, which are Archimedean. In particular an ordered Banach space with closed cone is Archimedean (see Theorem 2.1).

The following well-known key result seems to have been unnoticed.

Theorem 1.3. *Let (E, K, τ) be an ordered topological vector space; that is, (E, K) is an ordered vector space and (E, τ) is a topological vector space. Then, the following are equivalent:*

(i) $e \in \text{int}(K)$.
(ii) $[-e, e]$ *is a neighborhood of zero.*

In this case, e is an order unit.

A proof of this can be found in [1]; we present it here in the following form for the sake of completeness.

Proof. If $[-e, e]$ is a neighborhood of zero, then since

$$e + [-e, e] = [0, 2e] \subset K,$$

we have $e \in \text{int}(K)$. If $e \in \text{int}(K)$, then there is a circled neighborhood of zero V with $e + V \subset K$. Then $V \subset [-e, e]$. Indeed, for each $x \in V$ we have $-x \in V$, so $e + x$, $e - x \in K$. This shows that $x \in [-e, e]$. Hence $[-e, e]$ is a neighborhood of zero. □

We are now in a position to prove that there is no difference between metric and cone metric spaces via the following simple observation, whose proof is straightforward and is therefore omitted.

Theorem 1.4. *Let X be a non-empty set, E be an ordered vector space with cone K, and $e \in \text{int}(K)$. For a function $d : X \times X \to K$, define $\bar{d} : X \times X \to \mathbb{R}^+$ by*

$$\bar{d}(x,y) := \inf\{\lambda\colon\ d(x,y) \leqslant \lambda e\}.$$

(i) *If d is a cone metric, then $\bar{d}$ is a metric.*
(ii) *If $d : X \times X \to \mathbb{R}^+$ is a metric, then there exists a cone metric $p : X \times X \to K$ such that $d = \bar{p}$.*

2. Metrizability of cone metric spaces

As usual, the open ball of a metric space (X, d) with center $x \in X$ and radius $r > 0$ is denoted by $B_d(x, r)$ or, for short, by $B(x, r)$.